Semiclassical Analysis
Lecture 1

Semyon Dyatlov

August 23, 2018

Practical information

- Office hours: Tuesday 2–3 PM and by appointment, in 805 Evans
- Grading: I will assign several homework sets. Any math graduate student who submits solutions to enough homeworks will get an A
- Book: Maciej Zworski, Semiclassical Analysis, AMS, 2012
- Website: `http://math.berkeley.edu/~dyatlov/279/`
- Today's lecture is about motivation and pictures/movies. The formal definitions and a lot more explanations will come in later lectures. So don't be scared if you don't follow all the math – this is what the rest of the course is for!

Overview of today's lecture

One of the main concepts of semiclassical analysis is microlocalization, localization of functions in both position and frequency:

- Pseudodifferential operators, a generalization of multiplication operators: instead of $a(x)u(x)$ take $b(x, \frac{h}{i}\partial_x)u(x)$. This class includes differential operators and Fourier multipliers
- Wavefront set, a generalization of support: for $u = u(x; h) \in L^2(\mathbb{R}^n)$, we have $\mathrm{WF}_h(u) \subset \mathbb{R}^n_x \times \mathbb{R}^n_\xi$
- Here $h > 0$ is the semiclassical parameter, which is the wavelength (1/frequency) at which we study the function. We will work in the high frequency limit $h \to 0$, with remainders of the form $\mathcal{O}(h^N)$

Today I will show you 3 applications illustrated by numerics:

- Schrödinger evolution
- Quantum harmonic oscillator
- Quantum Ergodicity

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- Schrödinger evolution
- Quantum harmonic oscillator
- Quantum Ergodicity

Example 1: Schrödinger evolution

Schrödinger equation on $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$:

$$ih\partial_t u(t,x) + h^2\partial_x^2 u(t,x) = 0, \quad u|_{t=0} = u_0$$

Interpretation: $u =$ wavefunction of a quantum particle

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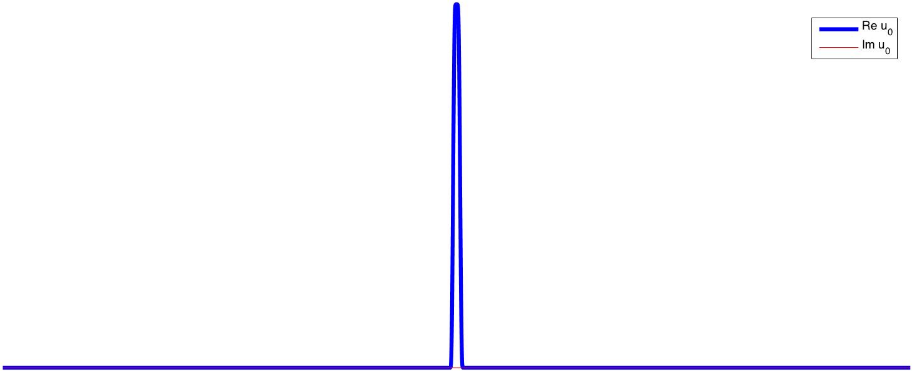
$$\text{Case 1: } u_0(x) = \chi(x/h), \quad \chi \in C_c^\infty((-1,1))$$



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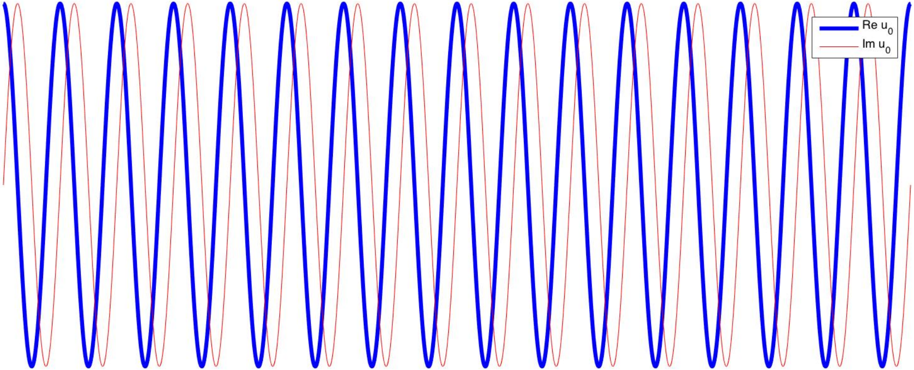

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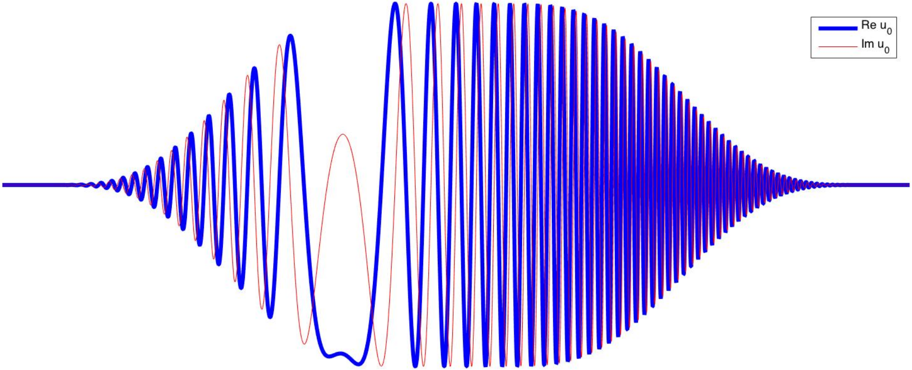

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Wavefront set

The picture becomes much clearer if we study concentration of u both in position and in frequency/Fourier space. We use the following

Definition [TO BE EXPLAINED IN THE COURSE]

Let $u = u(x;h) \in L^2(\mathbb{R})$ depend on $h > 0$. Define the wavefront set $\mathrm{WF}_h(u) \subset \mathbb{R}^2_{x,\xi}$ as follows: $(x_0, \xi_0) \notin \mathrm{WF}_h(u)$ iff there exist $\chi \in C_c^\infty(\mathbb{R})$, $\chi(x_0) \neq 0$ and $U \subset \mathbb{R}$ open, $\xi_0 \in U$ such that

$$\widehat{\chi u}(\xi/h) = \mathcal{O}(h^\infty), \quad \xi \in U$$

where $\mathcal{O}(h^\infty)$ means $\mathcal{O}(h^N)$ for all N

One way to numerically see the wavefront set is via the FBI transform:

$$\mathcal{T}_h u(x,\xi) = \int_{\mathbb{R}} e^{-\frac{i}{h}\langle y,\xi\rangle} e^{-\frac{|x-y|^2}{2h}} u(y)\,dy$$

$$(x_0,\xi_0) \notin \mathrm{WF}_h(u) \iff \mathcal{T}_h u(x,\xi) = \mathcal{O}(h^\infty) \quad \text{for} \quad (x,\xi) \quad \text{near} \quad (x_0,\xi_0)$$

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$$ih\partial_t u + h^2\partial_x^2 u = 0, \quad u|_{t=0} = u_0 \quad \Longrightarrow \quad u(t,\bullet) = e^{-itP/h}u_0$$

$$P = -h^2\partial_x^2 = \mathrm{Op}_h(p), \quad p(x,\xi) = \xi^2, \quad \mathrm{Op}_h(p) = p(x, \tfrac{h}{i}\partial_x)$$

Hamiltonian flow $e^{tH_p} : \mathbb{R}^2 \to \mathbb{R}^2$ generated by the vector field

$$H_p = (\partial_\xi p)\partial_x - (\partial_x p)\partial_\xi$$

For $p = \xi^2$, get $H_p = 2\xi\partial_x$, giving the ODE

$$\dot{x} = 2\xi, \quad \dot{\xi} = 0 \quad \Longrightarrow \quad e^{tH_p}(x,\xi) = (x + 2t\xi, \xi)$$

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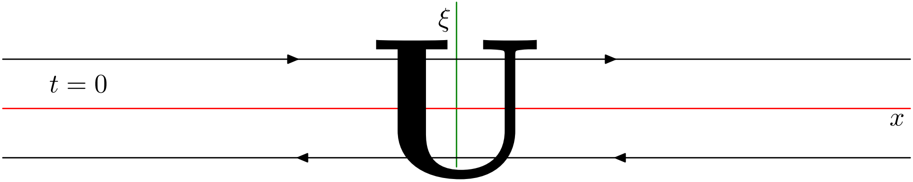

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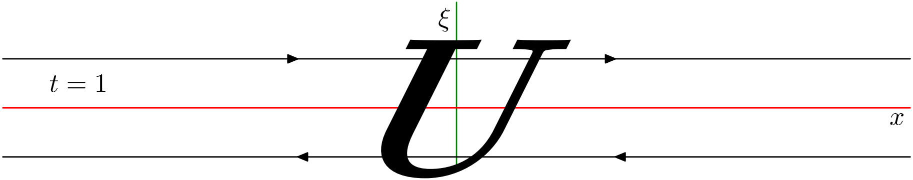

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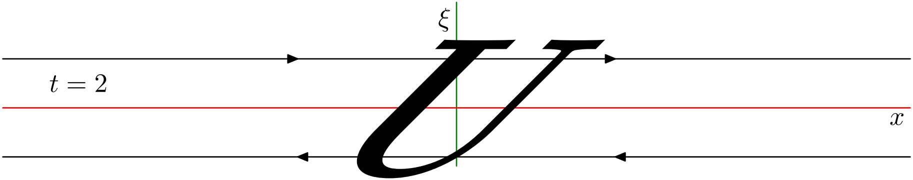

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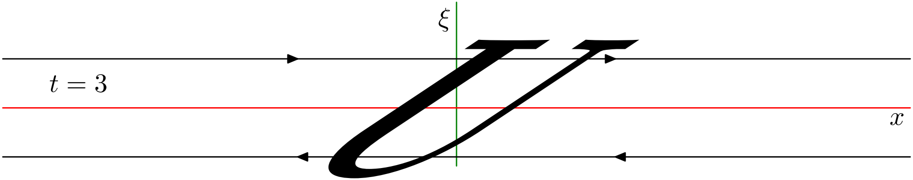

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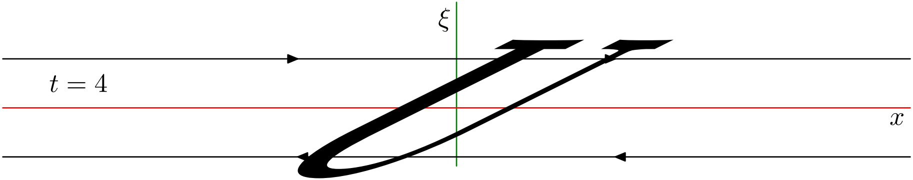

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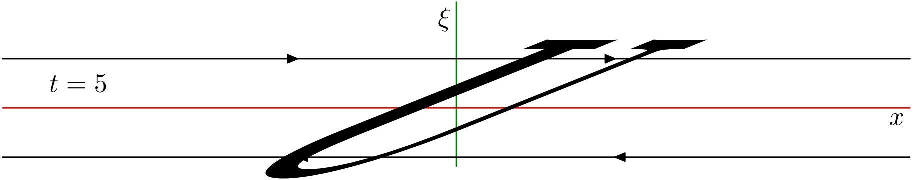

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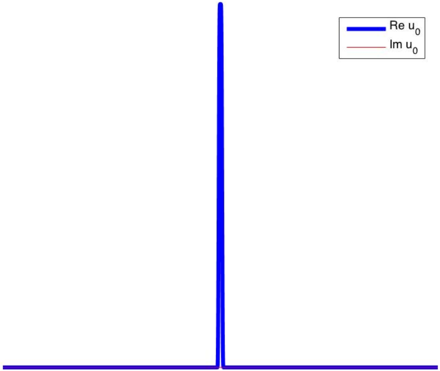

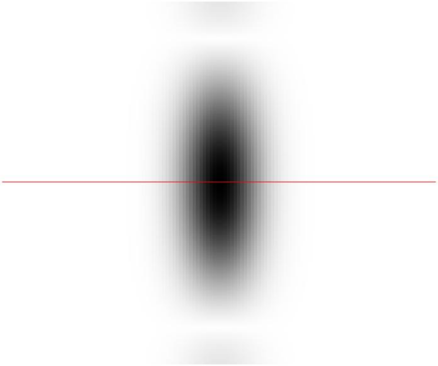

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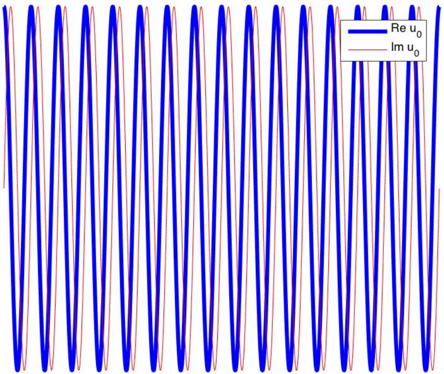

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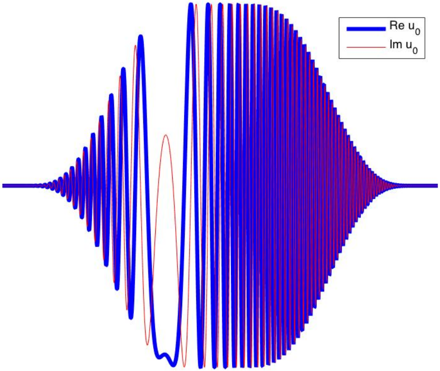

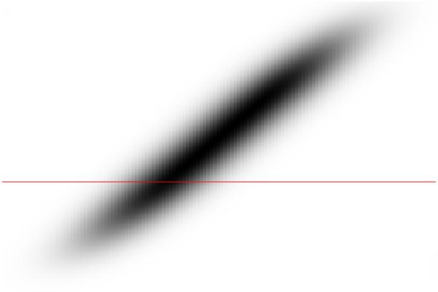

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Example 2: quantum harmonic oscillator

Classical harmonic oscillator: particle in potential field $V(x) = x^2$

$$p(x, \xi) = \xi^2 + x^2, \quad (x, \xi) \in \mathbb{R}^2$$

Quantum harmonic oscillator:

$$P(h) = \mathrm{Op}_h(p) = p(x, \tfrac{h}{i}\partial_x) = -h^2\partial_x^2 + x^2$$

Essentially self-adjoint on $L^2(\mathbb{R})$ with complete set of eigenfunctions

$$P(h)u_k = (2k+1)hu_k, \quad u_k(x) = Q_k(x/\sqrt{h})e^{-\frac{x^2}{2h}}, \quad k \geq 0$$

where $Q_k(x)$ is the k-th Hermite polynomial:

$$u_0(x) = e^{-\frac{x^2}{2h}}, \quad u_1(x) = \frac{x}{\sqrt{h}}e^{-\frac{x^2}{2h}}, \quad u_2(x) = (\tfrac{x^2}{h} - 1)e^{-\frac{x^2}{2h}}, \ldots$$

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Example 2: quantum harmonic oscillator

Classical harmonic oscillator: particle in potential field $V(x) = x^2$

$$p(x, \xi) = \xi^2 + x^2, \quad (x, \xi) \in \mathbb{R}^2$$

Quantum harmonic oscillator:

$$P(h) = \mathrm{Op}_h(p) = p(x, \tfrac{h}{i}\partial_x) = -h^2\partial_x^2 + x^2$$

Essentially self-adjoint on $L^2(\mathbb{R})$ with complete set of eigenfunctions

$$P(h)u_k = (2k+1)hu_k, \quad u_k(x) = Q_k(x/\sqrt{h})e^{-\frac{x^2}{2h}}, \quad k \geq 0$$

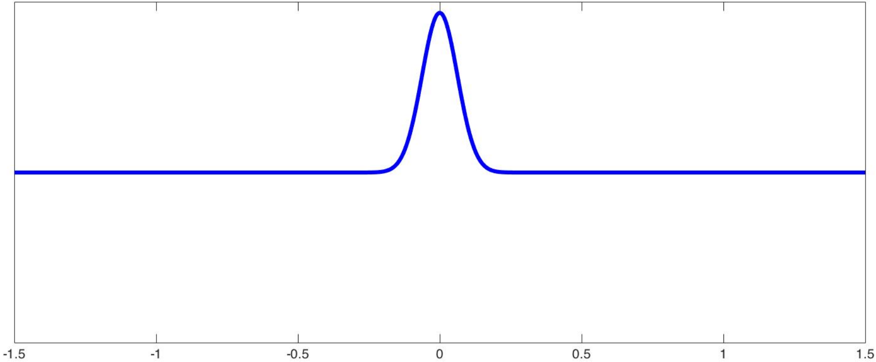
where $Q_k(x)$ is the k-th Hermite polynomial:

$$u_0(x) = e^{-\frac{x^2}{2h}}, \quad u_1(x) = \frac{x}{\sqrt{h}}e^{-\frac{x^2}{2h}}, \quad u_2(x) = (\frac{x^2}{h} - 1)e^{-\frac{x^2}{2h}}, \dots$$

Excited states of the quantum harmonic oscillator

$$P(h)u_k = (2k+1)hu_k.$$

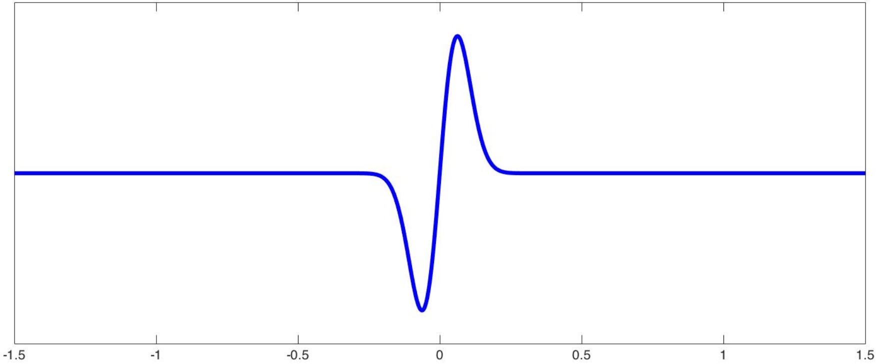
$$h = \frac{1}{256}, \quad k = 0, \quad u_k(x) = e^{-\frac{x^2}{2h}}$$



Excited states of the quantum harmonic oscillator

$$P(h)u_k = (2k+1)hu_k.$$

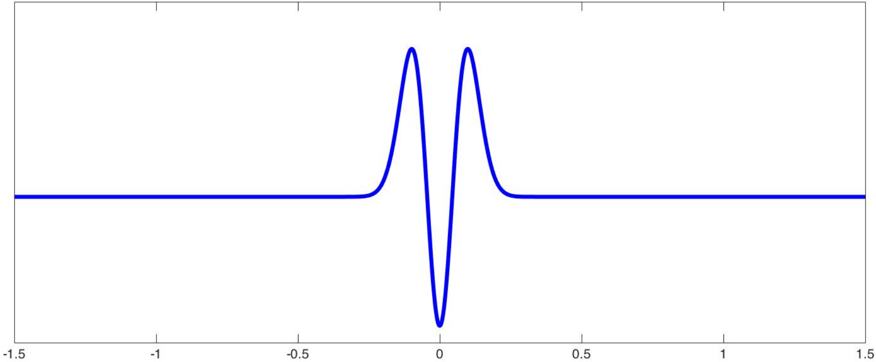
$$h = \tfrac{1}{256}, \quad k = 1, \quad u_k(x) = \tfrac{x}{\sqrt{h}} e^{-\frac{x^2}{2h}}$$



Excited states of the quantum harmonic oscillator

$$P(h)u_k = (2k+1)hu_k.$$

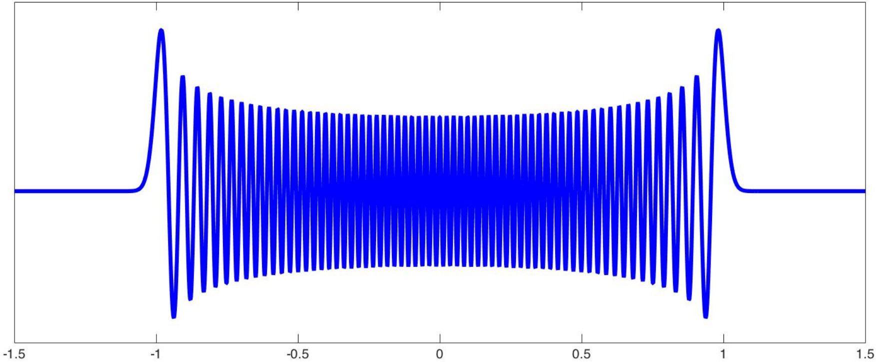
$$h = \tfrac{1}{256}, \quad k = 2, \quad u_k(x) = (\tfrac{x^2}{h} - 1)e^{-\frac{x^2}{2h}}$$



Excited states of the quantum harmonic oscillator

$$P(h)u_k = (2k+1)hu_k. \quad \text{Let} \quad (2k+1)h \approx 1 \quad \text{e.g.} \quad h = \frac{1}{2k} \ll 1$$

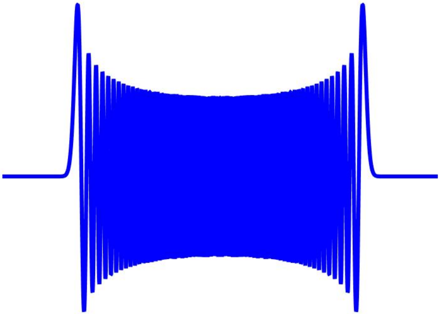
$$h = \frac{1}{256}, \quad k = 128, \quad u_k(x) = P_{128}\left(\frac{x}{\sqrt{h}}\right)e^{-\frac{x^2}{2h}}$$

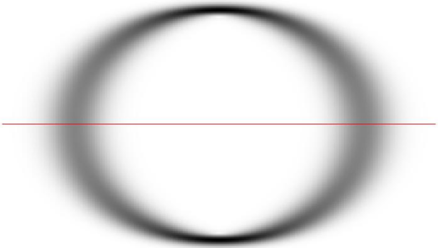


Excited states of the quantum harmonic oscillator

$P(h)u_k = (2k+1)hu_k$. Let $(2k+1)h \approx 1$ e.g. $h = \frac{1}{2k} \ll 1$

$$\mathrm{WF}_h(u_k) \subset \{p = 1\} = \{x^2 + \xi^2 = 1\}$$





Example 3: Quantum Ergodicity

- $M \subset \mathbb{R}^n$ bounded domain
- $-\Delta \geq 0$ Dirichlet Laplacian on M
- A sequence of eigenfunctions:

$$(-\Delta - \lambda_j^2)u_j = 0, \quad \lambda_j \xrightarrow[j\to\infty]{} \infty, \quad \|u_j\|_{L^2(M)} = 1$$

Question: Do $|u_j|^2$ equidistribute, i.e.

$$\int_M a(x)|u_j(x)|^2\,dx \to \frac{1}{\operatorname{vol}(M)} \int_M a(x)\,dx \quad \text{for all} \quad a \in C^\infty(M)?$$

Generalizations

- (M, g) Riemannian manifold (possibly with boundary)
- Microlocal equidistribution: replace $\int_M a(x)|u_j(x)|^2\,dx = \langle au, u\rangle_{L^2(M)}$ with $\langle \operatorname{Op}_h(b)u, u\rangle_{L^2(M)}$

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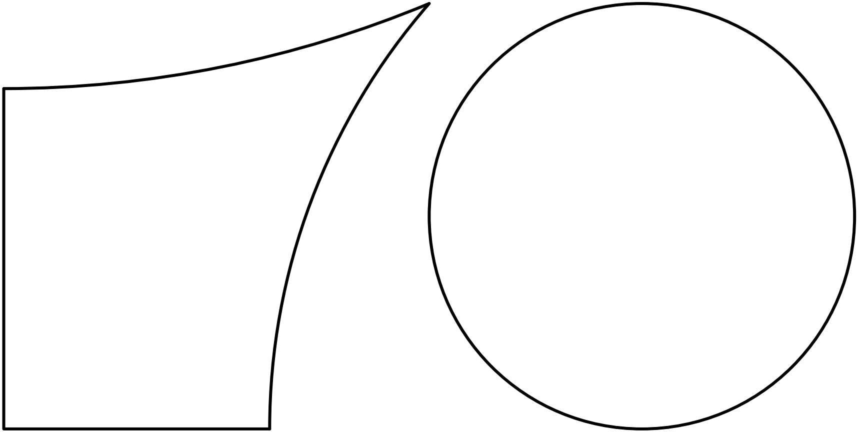
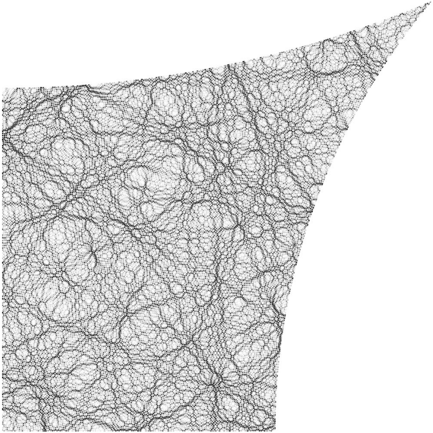
An example: two planar domains



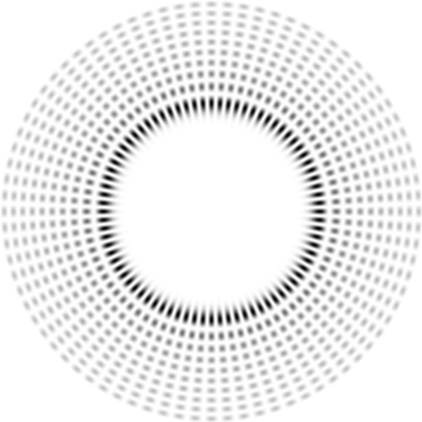
An example: two planar domains

Eigenfunction concentration

(picture on the left by Alex Barnett)



Equidistribution



No equidistribution

An example: two planar domains

Billiard ball dynamics

Chaotic

Completely integrable

$$(-\Delta - \lambda_j^2)u_j = 0, \quad \lambda_j \xrightarrow[j\to\infty]{} \infty, \quad \|u_j\|_{L^2(M)} = 1$$

Semiclassical reformulation: $(-h_j^2\Delta - 1)u_j = 0, \quad h_j := \lambda_j^{-1}$

Quantum Ergodicity [Shnirelman '74, Zelditch '87, Colin de Verdière '85 . . . Zelditch–Zworski '96]

Assume that the billiard ball flow on M is ergodic, i.e. all flow-invariant sets have zero Lebesgue measure or full measure. Then there exists a density 1 sequence of eigenfunctions $\{\lambda_{j_k}\}$ such that u_{j_k} equidistribute.

Generalizations

- (M, g) Riemannian manifold: use the geodesic flow
- Microlocal equidistribution w.r.t. the Liouville measure μ_L:

$$\langle \mathrm{Op}_{h_j}(b)u_j, u_j\rangle_{L^2(M)} \to \int_{S^*M} b\, d\mu_L \quad \text{for all} \quad b \in C^\infty(T^*M)$$

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Quantum Ergodicity gives a density 1 sequence of eigenfunctions which equidistribute. What about the rest? An active topic of study in quantum chaos with many results but the ultimate question (Quantum Unique Ergodicity conjecture of Rudnick–Sarnak) is still widely open…

Pictures by Alex Barnett

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Pictures by Alex Barnett