MATH 279 HOMEWORK 7

1. Fix $\psi \in C_c^\infty(\mathbb{R}^n)$.

(a) Let $y(h) \in \mathbb{R}^n$ satisfy $y(h) \to \infty$ as $h \to 0$ and define $u(h) \in L^2(\mathbb{R}^n)$ by

$$u(x;h) := \psi(x - y(h)).$$

Show that for every $a \in C_c^\infty(\mathbb{R}^{2n})$ we have

$$\langle a^{\mathrm{w}}(x, hD_x)u(h), u(h)\rangle_{L^2(\mathbb{R}^n)} \to 0 \quad \text{as} \quad h \to 0.$$

(Hint: fix $\chi \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$ such that $\chi\psi = \psi$ and define $\varphi(x;h) := \chi(x - y(h))$. Considering φ as a function of $(x, \xi; h)$ in the class $S(1)$, let φ^{w} be the corresponding quantization, which here is a multiplication operator. Apply the composition formula to the product $\varphi^{\mathrm{w}} a^{\mathrm{w}} \varphi^{\mathrm{w}}$ and use that $\varphi^{\mathrm{w}} u = u$. There is also a more direct solution using repeated integration by parts.)

(b) Let $\eta(h) \in \mathbb{R}^n$ satisfy $\eta(h) \to \infty$ as $h \to 0$ and define $v(h) \in L^2(\mathbb{R}^n)$ by

$$v(x;h) = e^{\frac{i}{h}\langle x, \eta(h)\rangle}\psi(x).$$

Show that for every $a \in C_c^\infty(\mathbb{R}^{2n})$ we have

$$\langle a^{\mathrm{w}}(x, hD_x)v(h), v(h)\rangle_{L^2(\mathbb{R}^n)} \to 0 \quad \text{as} \quad h \to 0.$$

(Hint: use the strategy of part (a) where φ^{w} should now be a Fourier multiplier, with $\varphi(x, \xi; h) = \chi(\xi - \eta(h))$ and $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 near 0.)

2. Let $P(h) := -h^2\Delta + V(x)$ where V is a potential satisfying the assumptions from the lectures. Assume that $E_0 \in \mathbb{R}$ satisfies $E_0 \geq \min V$. Using the Weyl Law, show that there exists a family of eigenvalues $E(h)$ of $P(h)$, $0 < h < h_0$, such that $E(h) \to E_0$ as $h \to 0$.

3. Let $P(h)$ as before and take $q \in S(1)$. Assume that $h_j \to 0$ and $u_j \in L^2(\mathbb{R}^n)$ satisfy as $j \to \infty$

$$\|(P(h_j) + hq^{\mathrm{w}}(x, h_j D_x))u_j\|_{L^2} = o(h_j), \quad \|u_j\|_{L^2} = 1.$$

Assume also that u_j converge weakly to a measure μ on $\mathbb{R}^{2n}$.

(a) Show that for all $b \in C_c^\infty(\mathbb{R}^{2n})$ we have

$$\int_{\mathbb{R}^{2n}} H_p b + 2(\operatorname{Im} q) b \, d\mu = 0.$$

(b) Assume that $P(h) = -h^2\partial_x^2 + |x|^2 - 1$ is the (shifted) one-dimensional quantum harmonic oscillator. Show that the integral of $\operatorname{Im} q$ on the unit circle in $\mathbb{R}^2$ is equal to 0 and μ has a C^∞ density (with respect to the standard measure on the circle) given by

e^F where F is a function on the circle such that $H_p F = 2\,\mathrm{Im}\,q$. (Hint: such F always exists locally, and $H_p b + 2(\mathrm{Im}\,q)b = e^{-F} H_p(e^F b)$.)