Math 1B, Section 106 Your name: ______________________

Quiz 2, September 9, 2009

Please write your name **on each sheet**. Show your work clearly and in order, including the intermediate steps in the solutions and the final answer.

Section 105

1. (3 pt) Compute the definite integral $\displaystyle\int_2^3 \frac{dx}{x^2-1}$.
2. (3 pt) Find the indefinite integral $\displaystyle\int \frac{dx}{x^2\sqrt{16-x^2}}$. Simplify your answer so that it does not include trigonometric functions.
3. (4 pt) Find the indefinite integral $\displaystyle\int \frac{dx}{1+\cos x}$.

Section 106

1. (3 pt) Find the indefinite integral $\displaystyle\int \frac{dx}{x^2\sqrt{25-x^2}}$. Simplify your answer so that it does not include trigonometric functions.
2. (3 pt) Compute the definite integral $\displaystyle\int_1^2 \frac{dx}{x^2+x}$.
3. (4 pt) Find the indefinite integral $\displaystyle\int \frac{dx}{1-\cos x}$.

Solutions for section 105

1. Since $x^2 - 1 = (x-1)(x+1)$, we have

$$\frac{1}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}$$

for some constants A and B. To find these, multiply both sides by $x^2 - 1$:

$$1 = A(x-1) + B(x+1) = (A+B)x + (B-A).$$

This gives us the system of equations

$$\begin{aligned} 0 &= A + B, \\ 1 &= B - A. \end{aligned}$$

The first equation yields $B = -A$; using this in the second equation, we get $1 = -2A$; thus $A = -\frac{1}{2}$ and $B = -A = \frac{1}{2}$. We can now find the indefinite integral:

$$\int \frac{dx}{x^2-1} = \int \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx = \frac{1}{2}(\ln|x-1| - \ln|x+1|) + C.$$

Finally, by the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_2^3 \frac{dx}{x^2-1} &= \frac{1}{2}(\ln|x-1| - \ln|x+1|)\Big|_{x=2}^3 \\ &= \frac{1}{2}(\ln 2 - \ln 4 - \ln 1 + \ln 3) = \frac{1}{2}\ln\left(\frac{3}{2}\right). \end{aligned}$$

2. Make the substitution $x = 4\sin\theta$, where $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$; then $dx = 4\cos\theta\, d\theta$, $\sqrt{16 - x^2} = 4\cos\theta$, and

$$\begin{aligned} \int \frac{dx}{x^2\sqrt{16-x^2}} &= \int \frac{4\cos\theta\, d\theta}{(4\sin\theta)^2(4\cos\theta)} \\ &= \frac{1}{16}\int \frac{d\theta}{\sin^2\theta} = -\frac{1}{16}\cot\theta + C. \end{aligned}$$

Now, $\cot\theta = \cos\theta/\sin\theta$. We know that $\sin\theta = x/4$ and we can find $\cos\theta = \sqrt{1-\sin^2\theta} = \frac{\sqrt{16-x^2}}{4}$; therefore,

$$\int \frac{dx}{x^2\sqrt{16-x^2}} = -\frac{\sqrt{16-x^2}}{16x} + C.$$

3. (Solution 1) Using the double angle identities and the substitution $t = \frac{x}{2}$, we get

$$\begin{aligned} \int \frac{dx}{1+\cos x} &= \int \frac{dx}{2\cos^2\frac{x}{2}} = \int \frac{dt}{\cos^2 t} \\ &= \tan t + C = \tan(x/2) + C. \end{aligned}$$

3. (Solution 2) Multiply the denominator and the numerator by $1 - \cos x$:

$$\int \frac{dx}{1+\cos x} = \int \frac{1-\cos x \, dx}{1-\cos^2 x} = \int \frac{1-\cos x \, dx}{\sin^2 x}$$
$$= \int \frac{dx}{\sin^2 x} - \int \frac{\cos x \, dx}{\sin^2 x} = -\cot x + \csc x + C$$
$$= \frac{1-\cos x}{\sin x} + C.$$

(We used the substitution $u = \sin x$ to evaluate the second of the two integrals above.) To see that the answer is the same as in the previous solution, use the double angle identities:

$$\frac{1-\cos x}{\sin x} = \frac{2\sin^2 \frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \tan\frac{x}{2}.$$

3. (Solution 3) Use the universal substitution from Stewart, Exercise 7.4.57:

$$t = \tan\frac{x}{2}, \quad dx = \frac{2\,dt}{1+t^2},$$
$$\cos x = \frac{1-t^2}{1+t^2}, \quad 1+\cos x = \frac{2}{1+t^2}.$$

We then get

$$\int \frac{dx}{1+\cos x} = \int dt = t + C = \tan\frac{x}{2} + C.$$

Solutions for section 106

1. Make the substitution $x = 5\sin\theta$, where $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$; then $dx = 5\cos\theta\,d\theta$, $\sqrt{25-x^2} = 5\cos\theta$, and

$$\int \frac{dx}{x^2\sqrt{25-x^2}} = \int \frac{5\cos\theta\,d\theta}{(5\sin\theta)^2(5\cos\theta)}$$
$$= \frac{1}{25}\int \frac{d\theta}{\sin^2\theta} = -\frac{1}{25}\cot\theta + C.$$

Now, $\cot\theta = \cos\theta/\sin\theta$. We know that $\sin\theta = x/5$ and we can find $\cos\theta = \sqrt{1-\sin^2\theta} = \frac{\sqrt{25-x^2}}{5}$; therefore,

$$\int \frac{dx}{x^2\sqrt{25-x^2}} = -\frac{\sqrt{25-x^2}}{25x} + C.$$

2. Since $x^2 + x = x(x+1)$, we have

$$\frac{1}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$$

for some constants A and B. To find these, multiply both sides by $x(x+1)$:

$$1 = A(x+1) + Bx = (A+B)x + A.$$

This gives us the system of equations

$$\begin{aligned} 0 &= A + B, \\ 1 &= A. \end{aligned}$$

From the second equation, we get $A = 1$; then we use the first equation to get $B = -A = -1$. We can now find the indefinite integral:

$$\int \frac{dx}{x^2+x} = \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \ln|x| - \ln|x+1| + C.$$

Finally, by the Fundamental Theorem of Calculus

$$\begin{aligned} \int_1^2 \frac{dx}{x^2+x} &= (\ln|x| - \ln|x+1|)\big|_{x=1}^2 \\ &= \ln 2 - \ln 3 - \ln 1 + \ln 2 = \ln\left(\frac{4}{3}\right). \end{aligned}$$

3. (Solution 1) Using the double angle identities and the substitution $t = \frac{x}{2}$, we get

$$\begin{aligned} \int \frac{dx}{1-\cos x} &= \int \frac{dx}{2\sin^2\frac{x}{2}} = \int \frac{dt}{\sin^2 t} \\ &= -\cot t + C = -\cot(x/2) + C. \end{aligned}$$

3. (Solution 2) Multiply the denominator and the numerator by $1 + \cos x$:

$$\begin{aligned} \int \frac{dx}{1-\cos x} &= \int \frac{1+\cos x\, dx}{1-\cos^2 x} = \int \frac{1+\cos x\, dx}{\sin^2 x} \\ &= \int \frac{dx}{\sin^2 x} + \int \frac{\cos x\, dx}{\sin^2 x} = -\cot x - \csc x + C \\ &= -\frac{1+\cos x}{\sin x} + C. \end{aligned}$$

(We used the substitution $u = \sin x$ to evaluate the second of the two integrals above.) To see that the answer is the same as in the previous solution, use the double angle identities:

$$\frac{1+\cos x}{\sin x} = \frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \cot\frac{x}{2}.$$

3. (Solution 3) Use the universal substitution from Stewart, Exercise 7.4.57:

$$\begin{aligned} t = \tan\frac{x}{2},\ \ dx = \frac{2\,dt}{1+t^2}, \\ \cos x = \frac{1-t^2}{1+t^2},\ \ 1-\cos x = \frac{2t^2}{1+t^2}. \end{aligned}$$

We then get

$$\int \frac{dx}{1+\cos x} = \int \frac{dt}{t^2} = -\frac{1}{t} + C = -\cot\frac{x}{2} + C.$$