Math 1B worksheet

Sep 23, 2009

Please split into groups of 2–4 people and solve the problems on the board. Please write the solutions as clearly as possible. You may pick the order in which to do the problems.

1–9. Compute the following limits (finite or infinite) or prove that they do not exist:

$$\lim_{n\to\infty} \arctan n, \tag{1}$$

$$\lim_{n\to\infty} \sqrt{\frac{n+1}{9n+1}}, \tag{2}$$

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right), \tag{3}$$

$$\lim_{n\to\infty} \frac{1}{n!}, \tag{4}$$

$$\lim_{n\to\infty} \frac{\sin n}{n}, \tag{5}$$

$$\lim_{n\to\infty} \cos \pi n, \tag{6}$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n, \tag{7}$$

$$\lim_{n\to\infty} \sqrt{n^2+n} - n, \tag{8}$$

$$\lim_{n\to\infty} \frac{(-3)^n}{n!}. \tag{9}$$

10. Consider the recursive sequence given by the formulas

$$x_1 = 2,\ x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

(a) Prove that x_n is a decreasing sequence and that $x_n \geqslant \sqrt{2}$ for all n.

(b) Assuming that the limit $X = \lim_{n\to\infty} x_n$ exists and that $X > 0$, find X.

Hints and answers

1. We know that $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$; therefore, $\lim_{n\to\infty} \arctan n = \frac{\pi}{2}$.

2. Divide both the numerator and the denominator by n.

Answer: $\frac{1}{3}$.

3. We have to find the limit $\lim_{n\to\infty} f\left(\frac{1}{n}\right)$, where $f(x) = \frac{\sin x}{x}$. But $\lim_{n\to\infty} \frac{1}{n} = 0$, so this is the same as $\lim_{x\to 0} f(x) = 1$ by L'Hôpital's Rule.

4. Since $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$, we have $n! \geqslant n$. We also know that $\lim_{n\to\infty} \frac{1}{n} = 0$; since $0 \leqslant \frac{1}{n!} \leqslant \frac{1}{n}$, we may apply the Squeeze Theorem.

Answer: 0.

5. We know that $\frac{-1}{n} \leqslant \frac{\sin n}{n} \leqslant \frac{1}{n}$; apply the Squeeze Theorem.

Answer: 0.

6. We use that $\cos(\pi n) = (-1)^n$.

Answer: Does not exist.

7. We have $\left(1 + \frac{1}{n}\right)^n = f\left(\frac{1}{n}\right)$, where $f(x) = (1+x)^{1/x} = e^{g(x)}$. Here $g(x) = \frac{\ln(1+x)}{x}$ has limit 1 as $x \to 0$ by L'Hôpital's Rule.

Answer: e.

8. We have $\sqrt{n^2+n} - n = f\left(\frac{1}{n}\right)$, where $f(x) = \frac{\sqrt{1+x}-1}{x}$ and $\lim_{x\to 0} f(x)$ can be found, say, by L'Hôpital's Rule.

Answer: $\frac{1}{2}$.

9. We have $n! = 1 \cdot 2 \cdot 3 \cdots n$; replacing all the terms but the first two and the last one by 3, we get $n! \geqslant 2 \cdot 3^{n-3} \cdot n$ and thus $\left|\frac{(-3)^n}{n!}\right| \leqslant \frac{27}{n}$. Now, use the Squeeze Theorem.

Answer: 0.

10. (a) First of all, $x_n > 0$ for all n (using mathematical induction). Next, $x_n + \frac{2}{x_n} = \left(\sqrt{x_n} - \sqrt{\frac{2}{x_n}}\right)^2 + 2\sqrt{2}$; therefore, $x_{n+1} \geqslant \sqrt{2}$ for all n. Using this inequality, we can see that x_n is decreasing.

(b) Passing to the limit on both sides of the identity $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$ and using that $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = X$, we get $X = \frac{1}{2}\left(X + \frac{2}{X}\right)$. Solving this, we get $X = \sqrt{2}$.