Math 1B worksheet

Nov 9, 2009

1. Consider the following logistic equation with harvesting:

$$y' = y(4 - y) - 3. \tag{1}$$

(a) Find the general solution of the equation.

(b) Find the solution with $y(0) = 2$.

(c) For the solution in part (b), find when it is increasing and when it is decreasing. Compute the limit of $y(x)$ as $x \to +\infty$.

(d) Find all equilibrium points of the equation (the solutions that are constant in time).

2–5. Solve the following differential equations:

$$y' = x + y, \tag{2}$$

$$xy' - 2y = x^2, \tag{3}$$

$$y' - y = \sin x, \tag{4}$$

$$y' + y = e^{2x}. \tag{5}$$

Hints and answers

1. By separation of variables, we have

$$x = \int \frac{dy}{y(4-y)-3} = -\int \frac{dy}{(y-1)(y-3)} = \frac{1}{2}\ln\left|\frac{y-1}{y-3}\right| + C.$$

Therefore,

$$y = \frac{1-3\tilde{C}e^{2x}}{1-\tilde{C}e^{2x}},$$

where $\tilde{C}$ can be any constant (positive, negative, or zero). There is also the equilibrium solution $y = 3$. (The other equilibrium solution is $y = 1$ and it is given by $\tilde{C} = 0$.)

If we put $y(0) = 2$, then $\tilde{C} = -1$ and

$$y = \frac{1+3e^{2x}}{1+e^{2x}}.$$

We see now that the limit of $y(x)$ as $x \to +\infty$ is 3. The function $y(x)$ is (strictly) increasing iff $y' > 0$, which means $(y-1)(3-y) > 0$, which is always true for our solution.

2. Answer: $-x - 1 + Ce^{x}$.
3. Answer: $x^2 \ln x + Cx^2$.
4. Answer: $-\frac{1}{2}(\sin x + \cos x) + Ce^{x}$.
5. Answer: $\frac{1}{3}e^{2x} + Ce^{-x}$.