18.156, SPRING 2017, PROBLEM SET 6

In this problem set, feel free to skip the more tedious technical details as long as you know how to handle them.

1. Let g be a Riemannian metric on $\mathbb{R}^n$:

$$g = \sum_{j,k=1}^{n} g_{jk}(x)\,dx_j dx_k, \quad g_{jk} \in C^\infty(\mathbb{R}^n;\mathbb{R}),$$

where the matrix with coefficients $g_{jk}(x)$ is symmetric and positive definite for all x. We assume that $g_{jk}(x) = \delta_{jk}$ for large x, that is g is a compactly supported perturbation of the Euclidean metric.

(a) Denote by Δ_g the Laplace–Beltrami operator corresponding to g and put $P := -h^2\Delta_g$. Show that $P \in \Psi_h^2(\mathbb{R}^n)$ and compute the principal symbol $p := \sigma_h(P)$.

(b)* Denote by $\varphi_t = \exp(tH_p)$ the Hamiltonian flow of p, that is

$$\partial_t(a \circ \varphi_t)|_{t=0} = \{p, a\} \quad \text{for all } a \in C^\infty(\mathbb{R}^{2n}).$$

Show that φ_t is related to the geodesic flow of g as follows: if $(x(t), \xi(t)) = \varphi_t(x_0, \xi_0)$ for some $(x_0, \xi_0) \in \mathbb{R}^{2n}$, then $t \mapsto x(t)$ is a geodesic and

$$2\xi_j(t) = \sum_{k=1}^{n} g_{jk}(x(t))\dot{x}_k(t).$$

2. Find the semiclassical principal symbols $\sigma_h(A)$ of the following operators in Ψ_h^2:

$$A = -h^2\Delta - 1 \quad \text{on } \mathbb{R}^n, \tag{1}$$

$$A = ih\partial_t - h^2\Delta_x \quad \text{on } \mathbb{R}_t \times \mathbb{R}_x^n, \tag{2}$$

$$A = h^2\partial_t^2 - h^2\Delta_x \quad \text{on } \mathbb{R}_t \times \mathbb{R}_x^n. \tag{3}$$

Determine the elliptic set $\mathrm{ell}_h(A)$ in each case and find formulas for the Hamiltonian flow of the principal symbol $\sigma_h(A)$.

3.* Assume that $a \in S^k(T^*\mathbb{R}^n)$, that is it has an asymptotic expansion in powers of $|\xi|$ as discussed in class on April 6. Show that $\langle\xi\rangle^{-k}a(x,\xi) \in C^\infty(T^*\mathbb{R}^n)$ extends to a smooth function on the fiber-radial compactification $\overline{T}^*\mathbb{R}^n$.

4. (a) Assume that $a(x, \xi; h) \in S^k_{1,0,h}$ is elliptic everywhere in the following sense: for some $c > 0$ and all $(x, \xi) \in \mathbb{R}^{2n}$, $0 < h \le 1$

$$|a(x, \xi; h)| \ge c\langle\xi\rangle^k. \tag{4}$$

Show that $1/a \in S^{-k}_{1,0,h}$.

(b)* Assume that (4) holds and $a \in S^k_h$, that is it has an asymptotic expansion in powers of h and $|\xi|$ as discussed in class on April 6. Show that $1/a \in S^{-k}_h$.

5. Consider the differential operator $P := h\partial_x + 1$ on $\mathbb{R}$. Show that P is elliptic everywhere, so that $\mathrm{ell}_h(P) = \overline{T}^*\mathbb{R}$. Applying the elliptic estimate, show that for each $\chi_0 \in C^\infty_c(\mathbb{R})$ there exists $\chi \in C^\infty_c(\mathbb{R})$ such that

$$\|\chi_0 u\|_{L^2} = \mathcal{O}(h^\infty)\|\chi u\|_{L^2} \quad \text{for all } u \in \mathcal{D}'(\mathbb{R}), \quad Pu = 0. \tag{5}$$

Find a nontrivial solution to the equation $Pu = 0$ and explain why (5) holds for it.