18.156, SPRING 2017, PROBLEM SET 4, SOLUTIONS

1 (a) Consider the operator $\widetilde{A}(\lambda) : \mathcal{H}_1 \oplus \mathbb{C}^N \to \mathcal{H}_2 \oplus \mathbb{C}^N$ defined as follows: $\widetilde{A}(\lambda)(f, \alpha) = (g, \beta)$ with

$$A(\lambda)f + \sum_{j=1}^{N} \alpha_j \partial_\lambda A(\lambda_0) u_j = g, \tag{1}$$

$$\langle \partial_\lambda A(\lambda_0) f, v_k \rangle_{\mathcal{H}_2} = \beta_k. \tag{2}$$

We make the following observations regarding the equations (1), (2) in the case $\lambda = \lambda_0$:

- Since $\langle A(\lambda_0) f, v_k\rangle_{\mathcal{H}_2} = \langle f, A(\lambda_0)^* v_k\rangle_{\mathcal{H}_1} = 0$ and $\langle \partial_\lambda A(\lambda_0) u_j, v_k\rangle_{\mathcal{H}_2} = \delta_{jk}$, we obtain
$$\alpha_j = \langle g, v_j\rangle_{\mathcal{H}_2}.$$
- In particular, if $g = 0$, then $\alpha = 0$ and $A(\lambda_0) f = 0$, implying that f is a linear combination of $u_1, \dots, u_N$. This implies
$$g = 0 \quad \Longrightarrow \quad f = \sum_{j=1}^{N} \beta_j u_j.$$
- We see that $\widetilde{A}(\lambda_0)$ has no kernel. Since $\widetilde{A}(\lambda_0)$ is Fredholm of index 0, it is invertible. Thus $\widetilde{A}(\lambda)^{-1}$ is invertible for λ near λ_0. Denote the inverse as follows:
$$\widetilde{A}(\lambda)^{-1} = \begin{pmatrix} B_{11}(\lambda) & B_{12}(\lambda) \\ B_{21}(\lambda) & B_{22}(\lambda) \end{pmatrix} : \mathcal{H}_2 \oplus \mathbb{C}^N \to \mathcal{H}_1 \oplus \mathbb{C}^N.$$

  The above observations imply that

$$B_{12}(\lambda_0)\beta = \sum_{j=1}^{N} \beta_j u_j, \quad \big(B_{21}(\lambda_0) g\big)_j = \langle g, v_j\rangle_{\mathcal{H}_2}, \quad B_{22}(\lambda_0) = 0. \tag{3}$$

Now, using the formula $\partial_\lambda(\widetilde{A}(\lambda)^{-1}) = -\widetilde{A}(\lambda)^{-1}\partial_\lambda A(\lambda)\widetilde{A}(\lambda)^{-1}$, we obtain from (3)

$$\partial_\lambda B_{22}(\lambda_0) = -B_{21}(\lambda_0)\partial_\lambda A(\lambda_0) B_{12}(\lambda_0) = -I. \tag{4}$$

In particular, we have the Laurent expansion of $B_{22}(\lambda)^{-1}$ (where $\dots$ denotes terms homomorphic at λ_0):

$$B_{22}(\lambda)^{-1} = -\frac{I}{\lambda - \lambda_0} + \dots$$

Using Schur's complement formula

$$A(\lambda)^{-1} = B_{11}(\lambda) - B_{12}(\lambda) B_{22}(\lambda)^{-1} B_{21}(\lambda),$$

we get the required Laurent expansion:

$$A(\lambda)^{-1} = \frac{B_{12}(\lambda_0)B_{21}(\lambda_0)}{\lambda - \lambda_0} + \dots$$

1 (b) First, assume that the matrix (a_{jk}) is invertible. Changing the basis $v_1, \dots, v_N$, we can make this matrix equal to the identity matrix, in which case $J = 1$ by part (a).

Now, assume that $J = 1$. Take $w \in \ker A(\lambda_0)^*$. Then

$$w = A(\lambda)A(\lambda)^{-1}w = \frac{A(\lambda_0)A_1 w}{\lambda - \lambda_0} + A(\lambda_0)A_0(\lambda_0)w + A'(\lambda_0)A_1 w + \mathcal{O}(\lambda - \lambda_0),$$

implying that

$$A(\lambda_0)A_1 w = 0, \quad A(\lambda_0)A_0(\lambda_0)w + A'(\lambda_0)A_1 w = w.$$

Pairing the second identity with w and using that $A(\lambda_0)^* w = 0$, we get

$$\langle A'(\lambda_0)A_1 w, w\rangle_{\mathcal{H}_2} = \|w\|^2_{\mathcal{H}_2}.$$

Thus for each $w \in \ker A(\lambda_0)^* \setminus \{0\}$ there exists $u := A_1 w \in \ker A(\lambda_0)$ such that $\langle A'(\lambda_0)u, w\rangle_{\mathcal{H}_2} \neq 0$. This implies that the matrix (a_{jk}) is invertible.

2. We have

$$f = (P_V - \lambda^2)u = (-\Delta - \lambda^2 + V)u = g + Vu \in L^2_{\mathrm{comp}}(\mathbb{R}^n).$$

To show that $u = R_V(\lambda)f$ it suffices to prove the identity

$$R_0(\lambda)g = R_V(\lambda)(P_V - \lambda^2)R_0(\lambda)g \quad \text{for all } g \in L^2_{\mathrm{comp}}(\mathbb{R}^n). \tag{5}$$

For $\operatorname{Im}\lambda > 0$ and λ not a resonance, (5) is immediate since $R_V(\lambda) = (P_V - \lambda^2)^{-1} : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$ and $R_0(\lambda)g \in H^2(\mathbb{R}^n)$. For general λ, (5) follows by analytic continuation, where to make sure that the right-hand side is well-defined we rewrite (5) as follows:

$$R_0(\lambda)g = R_V(\lambda)(g + VR_0(\lambda)g).$$

3. Taking $\rho \in C_c^\infty(\mathbb{R}^n)$ such that $\rho V = V$, we have the identity

$$R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}(I - VR_0(\lambda)(1-\rho)). \tag{6}$$

It follows that for each $f \in L^2_{\mathrm{comp}}(\mathbb{R}^n)$,

$$(I + VR_0(\lambda)\rho)^{-1}f = (-\Delta - \lambda^2)R_V(\lambda)(I + VR_0(\lambda)(1-\rho))f \in L^2_{\mathrm{comp}}. \tag{7}$$

Therefore, since $R_V(\lambda)$ has a semisimple pole at λ_0, so does $(I + VR_0(\lambda)\rho)^{-1}$. We write (with $A_0(\lambda)$ holomorphic at λ_0)

$$(I + VR_0(\lambda)\rho)^{-1} = A_0(\lambda) + \frac{A_1}{\lambda^2 - \lambda_0^2} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n). \tag{8}$$

Note that A_1 maps $L^2_{\text{comp}}(\mathbb{R}^n) \to L^2_{\text{comp}}(\mathbb{R}^n)$ since $(I + VR_0(\lambda)\rho)^{-1}$ does. By writing out the Laurent expansion of (6) we have

$$\Pi_0 = R_0(\lambda_0)A_1(I - VR_0(\lambda_0)(1-\rho)).$$

Since $A_1(I - VR_0(\lambda)(1-\rho)) : L^2_{\text{comp}}(\mathbb{R}^n) \to L^2_{\text{comp}}(\mathbb{R}^n)$, the range of Π_0 consists of functions which are outgoing at λ_0. *Note that this would fail if the expansion (8) had higher negative powers of $\lambda - \lambda_0$, since then we would have to involve the derivatives $\partial_\lambda^j R_0(\lambda_0)$ for $j \geq 1$ and these do not give outgoing functions.*

Given the statement that $(P_V - \lambda_0^2)\Pi_0 = 0$ (which follows by writing out the Laurent expansion of the identity $(P_V - \lambda^2)R_V(\lambda) = I$), we see that

$$\Pi_0(L^2_{\text{comp}}(\mathbb{R}^n)) \subset \{u \in H^2_{\text{loc}}(\mathbb{R}^n) \mid (P_V - \lambda_0^2)u = 0,\ u \text{ is outgoing at } \lambda_0\}.$$

To show the opposite containment, assume that $u = R_0(\lambda_0)g$ for some $g \in L^2_{\text{comp}}(\mathbb{R}^n)$ and $(P_V - \lambda_0^2)u = 0$. Then $g = -VR_0(\lambda_0)g$, which also implies that $g = \rho g$. Then

$$(I + VR_0(\lambda)\rho)g = V(R_0(\lambda) - R_0(\lambda_0))g.$$

Take the following family of functions holomorphic in λ:

$$G(\lambda) = \frac{V(R_0(\lambda) - R_0(\lambda_0))g}{\lambda^2 - \lambda_0^2} \in L^2_{\text{comp}}(\mathbb{R}^n).$$

Taking the constant term in the Laurent expansion at $\lambda = \lambda_0$ of the identity

$$(I + VR_0(\lambda)\rho)^{-1}(\lambda^2 - \lambda_0^2)G(\lambda) = g,$$

we see that

$$g = A_1 G(\lambda_0).$$

Since $\rho G(\lambda_0) = G(\lambda_0)$, we get

$$\begin{aligned} u = R_0(\lambda_0)g &= R_0(\lambda_0)A_1(I - VR_0(\lambda_0)(1-\rho))G(\lambda_0) \\ &= \Pi_0 G(\lambda_0) \end{aligned}$$

and thus u lies in the range of Π_0 as required.

4. Take ρ as in Exercise 3. It follows from (7) that the singular part of the Laurent expansion of $(I + VR_0(\lambda)\rho)^{-1}$ at λ_0 only has powers $(\lambda - \lambda_0)^{-1}, \dots, (\lambda - \lambda_0)^{-J}$. Let A_J be the highest order term in this expansion, namely

$$(I + VR_0(\lambda)\rho)^{-1} = \frac{A_J}{(\lambda^2 - \lambda_0^2)^J} + \mathcal{O}\big((\lambda - \lambda_0)^{1-J}\big).$$

Then (6) gives

$$R_V(\lambda) = \frac{R_0(\lambda_0)A_J(I - VR_0(\lambda_0)(1-\rho))}{(\lambda^2 - \lambda_0^2)^J} + \mathcal{O}\big((\lambda - \lambda_0)^{1-J}\big).$$

It follows that

$$(P_V - \lambda_0^2)^{J-1}\Pi_{\lambda_0} = R_0(\lambda_0)A_J(I - VR_0(\lambda_0)(1-\rho)).$$

This implies that the range of $(P_V - \lambda_0^2)^{J-1}\Pi_{\lambda_0}$ consists of outgoing functions, and writing out the Laurent expansion of the identity $(P_V - \lambda^2)R_V(\lambda) = I$ at $\lambda = \lambda_0$ we see that functions in this range also solve $(P_V - \lambda_0^2)u = 0$.