18.156, SPRING 2017, PROBLEM SET 3, SOLUTIONS

1. Take arbitrary $x^0 \in \operatorname{supp} a$. Since $\nabla\Phi(x) \neq 0$, by the inverse mapping theorem there exists open sets $V_{x^0} \ni x^0$, W_{x^0} in $\mathbb{R}^n$ and a diffeomorphism $\psi_{x^0} : V_{x^0} \to W_{x^0}$ such that $\Phi = x_1 \circ \psi_{x^0}$ on V_{x^0}, where $x_1 : \mathbb{R}^n \to \mathbb{R}$ is the coordinate map.

The sets $\{V_{x^0} \mid x^0 \in \operatorname{supp} a\}$ form an open cover of $\operatorname{supp} a$. Take a finite subcover $V_1, \dots, V_m$ and let $\psi_j : V_j \to W_j$ be the corresponding diffeomorphisms. Take a partition of unity $\chi_1, \dots, \chi_m$ subordinate to the cover of $V_1, \dots, V_m$. Changing variables in the integral, we have

$$I(h) = \sum_{j=1}^m I_j(h), \quad I_j(h) := \int_{V_j} e^{i\Phi(x)/h} \chi_j(x) a(x)\, dx = \int_{W_j} e^{ix_1/h} a_j(x)\, dx$$

where $a_j = (\chi_j a) \circ \psi_j^{-1} \cdot J_j \in C_c^\infty(W_j)$ and J_j is the Jacobian of ψ_j^{-1}.

It remains to show that each $I_j(h)$ is $\mathcal{O}(h^\infty)$. For that, integrate by parts N times in x_1:

$$I_j(h) = \int_{W_j} \big((-ih\partial_{x_1})^N e^{ix_1/h}\big) a_j(x)\, dx = (ih)^N \int_{W_j} e^{ix_1/h} \partial_{x_1}^N a_j(x)\, dx$$

which gives

$$|I_j(h)| \leq C_{j,N} h^N, \quad C_{j,N} := \|\partial_{x_1}^N a_j\|_{L^1}.$$

2. We will show the stronger statement

$$\operatorname{WF}_h(u) \subset X := \{(x, \partial_x\varphi(x,\theta)) \mid (x,\theta) \in \operatorname{supp} a,\ \partial_\theta\varphi(x,\theta) = 0\}. \tag{1}$$

Assume that $(x_0, \xi_0) \notin X$. Choose $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $\chi(x_0) \neq 0$ and a small ball $W \subset \mathbb{R}^n$ centered at ξ_0 such that

$$X \cap (\operatorname{supp} \chi \times \overline{W}) = \emptyset. \tag{2}$$

We compute for $\xi \in \mathbb{R}^n$

$$\widehat{\chi u}(\xi/h) = \int_{\mathbb{R}^{n+m}} e^{i\Phi_\xi(x,\theta)/h} b_\chi(x,\theta)\, dx d\theta$$

where $\Phi_\xi \in C^\infty(U; \mathbb{R})$, $b_\chi \in C_c^\infty(U; \mathbb{C})$ are given by

$$\Phi_\xi(x,\theta) = \varphi(x,\theta) - \langle x, \xi\rangle, \quad b_\chi(x,\theta) = a(x,\theta)\chi(x).$$

We have

$$\partial_x\Phi_\xi(x,\theta) = \partial_x\varphi(x,\theta) - \xi, \quad \partial_\theta\Phi_\xi(x,\theta) = \partial_\theta\varphi(x,\theta).$$

By (2), for $\xi \in W$ the phase Φ_ξ has no stationary points on $\operatorname{supp} b_\chi$. Therefore, by Exercise 1

$$\widehat{\chi u}(\xi/h) = \mathcal{O}(h^\infty), \quad \xi \in W.$$

The latter statement is in fact uniform in $\xi \in W$, as can be seen by carefully examining the solution of Exercise 1. (Uniformity of nonstationary and stationary phase in a parameter, here ξ, is both true and very useful in semiclassical analysis, but is usually made implicit.) Therefore, we obtain

$$(x_0, \xi_0) \notin \mathrm{WF}_h(u)$$

which gives (1).

3 (a) We write

$$I_{xa}(h) = \int_{\mathbb{R}} x e^{ix^2/h} a(x)\,dx = -\frac{ih}{2}\int_{\mathbb{R}} \partial_x(e^{ix^2/h}) a(x)\,dx.$$

Integrating by parts (which is fine since a is Schwartz) we obtain

$$I_{xa}(h) = \frac{ih}{2}\int_{\mathbb{R}} e^{ix^2/h} a'(x)\,dx = \frac{ih}{2} I_{a'}(h).$$

Next, assume that $a(0) = 0$. Then we may write $a = xb$ where b is a Schwartz function. Indeed, the fact that $x^j \partial_x^k (a(x)/x)$ is bounded for large $|x|$ is verified directly, and to establish that $a(x)/x$ extends smoothly to $x = 0$ we use the representation

$$a(x) = xb(x), \quad b(x) = \int_0^1 a'(tx)\,dt.$$

Now we have

$$I_a(h) = I_{xb}(h) = \frac{ih}{2} I_{b'}(h) = \mathcal{O}(h).$$

3 (b) Define

$$F(s) = \int_{\mathbb{R}} e^{-sx^2}\,dx, \quad s \in \mathbb{C}, \quad \operatorname{Re} s > 0.$$

The integral converges exponentially fast and the integrated function is holomorphic in s, therefore $F(s)$ is holomorphic in s as well. For real $s > 0$, using change of variables $y = s^{1/2}x$ and the Gaussian integral we compute

$$F(s) = \sqrt{\frac{\pi}{s}}. \tag{3}$$

Since both sides are holomorphic in $\{\operatorname{Re} s > 0\}$, the formula (3) holds for all $\operatorname{Re} s > 0$. Here we choose the (usual) branch of the square root $\sqrt{z}$ on $\{\operatorname{Re} z > 0\}$ such that $\sqrt{1} = 1$. Now we compute for $a(x) = e^{-x^2}$,

$$I(h) = F\Big(1 - \frac{i}{h}\Big) = \sqrt{\frac{\pi h}{h - i}} = \sqrt{\pi} e^{i\pi/4} h^{1/2} + \mathcal{O}(h^{3/2}).$$

3 (c) We write

$$a = a(0)e^{-x^2} + b, \quad b(0) = 0.$$

From Exercise 3(a), we have $I_b(h) = \mathcal{O}(h)$. Using the formula from Exercise 3(b), we get

$$I_h(a) = h^{1/2} \cdot \sqrt{\pi} e^{i\pi/4} a(0) + \mathcal{O}(h).$$