18.156, SPRING 2017, PROBLEM SET 1, SOLUTIONS

1. Recall d'Alembert's formula for the wave operator $\Box_0 = \partial_t^2 - \partial_x^2$:

$$w(t,x) = \frac{w(0,x-t)+w(0,x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} \partial_t w(0,y)\,dy + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} \Box_0 w(s,y)\,dsdy, \quad t \geq 0.$$

In our case, $\Box_0 w = g - Vw$, therefore

$$\begin{aligned} w(t,x) &= \frac{f_0(x-t)+f_0(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} f_1(y)\,dy \\ &+ \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} g(s,y) - V(y)w(s,y)\,dsdy. \end{aligned} \tag{1}$$

1 (a) Fix $r_0 > 0$ such that $\operatorname{supp} V, \operatorname{supp} f_0, \operatorname{supp} f_1, \operatorname{supp} g \subset \{|x| < r_0\}$. Then (1) implies

$$\operatorname{supp} w \cap \{t \geq 0\} \subset \{|x| \leq r_0 + t\}.$$

1 (b) We find from (1) that $w(t,x) = w_\pm(x \mp t)$ for $t \geq 0$, $|x| \geq r_0$ where

$$\begin{aligned} w_+(x) &= \frac{f_0(x)}{2} + \frac{1}{2}\int_x^\infty f_1(y)\,dy + \frac{1}{2}\int_0^\infty \int_{x+s}^\infty g(s,y) - V(y)w(s,y)\,dyds, \\ w_-(x) &= \frac{f_0(x)}{2} + \frac{1}{2}\int_{-\infty}^x f_1(y)\,dy + \frac{1}{2}\int_0^\infty \int_{-\infty}^{x-s} g(s,y) - V(y)w(s,y)\,dyds. \end{aligned}$$

2 (a) Differentiating under the integral sign and integrating by parts (recall that by Exercise 1(a) the support of the integrand is compact) we compute

$$\mathcal{E}'(t) = \operatorname{Re} \int_{\mathbb{R}} \overline{w_t} w_{tt} + \overline{w_{xt}} w_x + V \overline{w_t} w\,dx = \operatorname{Re} \int_{\mathbb{R}} \overline{w_t} g\,dx$$

which gives the required identity for $\mathcal{E}(t)$. It follows that $\mathcal{E}(T)$ is constant for T large enough (specifically, as soon as $\operatorname{supp} g \subset \{t < T\}$).

Now, assume that $V \geq 0$. Then the quantity

$$\frac{1}{2}\int_{\mathbb{R}} |\partial_t w(t,x)|^2 + |\partial_x w(t,x)|^2\,dx \leq \mathcal{E}(t)$$

is bounded uniformly in t. It remains to show the bound

$$\int_{\mathbb{R}} |w(t,x)|^2\,dx \leq C(1+t)^2, \quad t \geq 0.$$

Recall that $\operatorname{supp} w(t,\bullet)$ is contained in an interval of size $2t+C$ by Exercise 1(a). Then by Poincaré inequality we have

$$\int_{\mathbb{R}} |w(t,x)|^2\,dx \leq C(1+t)^2 \int_{\mathbb{R}} |w_x(t,x)|^2\,dx \leq C(1+t)^2\mathcal{E}(t)$$

which finishes the proof.

2 (b) Put $C_V := \max(2, \sup|V-1|)$. We estimate

$$\begin{aligned}
\mathcal{E}_0'(t) &= \operatorname{Re}\int_{\mathbb{R}} \overline{w_t}w_{tt} + \overline{w_{xt}}w_x + \overline{w_t}w\,dx = \operatorname{Re}\int_{\mathbb{R}} \overline{w_t}(g-(V-1)w)\,dx\\
&\leq \frac{1}{2}\int_{\mathbb{R}} 2|w_t|^2 + |g|^2 + |V-1|\cdot|w|^2\,dx\\
&\leq C_V\mathcal{E}_0(t) + \frac{1}{2}\int_{\mathbb{R}} |g|^2\,dx.
\end{aligned}$$

It remains to use Gronwall's inequality and recall that g is compactly supported.

3. Define the energy quantity

$$\mathcal{E}_1(t) := \frac{1}{2}\int_{x_0-t_0+t}^{x_0+t_0-t} |w_t|^2 + |w_x|^2 + |w|^2\,dx, \quad 0 \leq t \leq t_0.$$

We compute

$$\begin{aligned}
\mathcal{E}_1'(t) = &-\frac{|w_t(t,x_0-t_0+t) + w_x(t,x_0-t_0+t)|^2 + |w(t,x_0-t_0+t)|^2}{2}\\
&-\frac{|w_t(t,x_0+t_0-t) - w_x(t,x_0+t_0-t)|^2 + |w(t,x_0+t_0-t)|^2}{2}\\
&+\operatorname{Re}\int_{x_0-t_0+t}^{x_0+t_0-t} \overline{w_t}(g-(V-1)w)\,dx.
\end{aligned}$$

Here we need to be careful because the limits of integration depend on t and integration by parts in x produces boundary terms. We then get as in Exercise 2(b)

$$\mathcal{E}_1'(t) \leq C_V\mathcal{E}_1(t) + \frac{1}{2}\int_{x_0-t_0+t}^{x_0+t_0-t} |g|^2\,dx.$$

However, then the vanishing condition on f_0, f_1, g implies that $\mathcal{E}_1'(0) = 0$ and $\mathcal{E}_1'(t) \leq C_V\mathcal{E}_1(t)$, which immediately gives $\mathcal{E}_1(t) = 0$ for all $t \in [0,t_0]$. This implies that $w(t,x) = 0$ almost everywhere for $0 \leq t \leq t_0$ and $|x-x_0| \leq t_0 - t$, which by continuity gives $w(t_0,x_0) = 0$.

4. We first deal with uniqueness. Assume that u solves

$$(P_V - \lambda^2)u = f; \quad u(x) \sim e^{\pm i\lambda x} \quad \text{for } \pm x \gg 1. \tag{2}$$

Then we have

$$\partial_x W(u, e_\pm) = e_\pm \cdot f; \quad W(u,e_\pm) = 0 \quad \text{for } \pm x \gg 1.$$

Therefore,

$$W(u,e_-)(x)=\int_{-\infty}^x e_-(y)f(y)\,dy,\quad W(u,e_+)(x)=-\int_x^\infty e_+(y)f(y)\,dy.$$

Using the identity

$$u=\frac{W(u,e_-)e_+-W(u,e_+)e_-}{\mathbf{W}(\lambda)}$$

we see that

$$u(x)=\int_{\mathbb{R}} R_V(x,y;\lambda)f(y)\,dy. \tag{3}$$

To show existence, fix f and define u by (3). Then it is straightforward to verify that u solves (2).

5. Denote $W_\pm := W(e_+,e^{\pm i\lambda x})$. Since $(P_V-\lambda^2)e^{\pm i\lambda x}=Ve^{\pm i\lambda x}$, we have

$$\partial_x W_\pm(x)=-V(x)e_+(x)e^{\pm i\lambda x}$$

and from the fact that $e_+(x)=e^{i\lambda x}$ for $x\gg 1$ we have

$$W_+(x)=0,\quad W_-(x)=-2i\lambda\quad\text{for } x\ge r_0.$$

Together these imply the required integral identities.

Next, choose $r_0>0$ such that $\operatorname{supp}V\subset[-r_0,r_0]$ and put $C_V:=e^{2C_0r_0}\sup|V|$. Using the identity

$$e_+(x)=\frac{i}{2\lambda}\big(W_-(x)e^{i\lambda x}-W_+(x)e^{-i\lambda x}\big)$$

and the fact that $|\operatorname{Im}\lambda|\le C_0$ we get we get the bound

$$\sup_x|V(x)e_+(x)e^{\pm i\lambda x}|\le\frac{C_V}{2|\lambda|}\big(|W_+|+|W_-|\big).$$

Therefore, for $|x|\le r_0$ we have

$$|W_+(x)|+|W_-(x)+2i\lambda|\le\frac{C_V}{|\lambda|}\int_x^{r_0}|W_+(y)|+|W_-(y)|\,dy,$$

which by Gronwall's inequality implies

$$|W_+(x)|+|W_-(x)+2i\lambda|\le 4C_Vr_0\exp\Big(\frac{2C_Vr_0}{|\lambda|}\Big)=\mathcal{O}(1) \tag{4}$$

for $|x|\le r_0$, and thus for all x since $W_\pm(x)$ are constant for $\pm x>r_0$. This gives the required asymptotics of $W_\pm$, which by the identity

$$\begin{pmatrix}e_+(x)\\ e_+'(x)\end{pmatrix}=\frac{i}{2\lambda}\begin{pmatrix}-e^{-i\lambda x} & e^{i\lambda x}\\ i\lambda e^{-i\lambda x} & i\lambda e^{i\lambda x}\end{pmatrix}\begin{pmatrix}W_+(x)\\ W_-(x)\end{pmatrix}$$

gives the required asymptotics on e_+,e_+'. The asymptotics of e_-,e_-' are proved similarly.

6 (a). The function $\mathbf{W}(\lambda)$ is holomorphic in $\lambda \in \mathbb{C}$. To show that $\mathbf{W}(\lambda)^{-1}$ is meromorphic it then suffices to prove that $\mathbf{W}(\lambda)$ is not identically zero. One way to see this is to use the asymptotic formulae for $e_\pm$ from Exercise 5, which imply

$$\mathbf{W}(\lambda) = -2i\lambda + \mathcal{O}(1), \quad |\operatorname{Im}\lambda| \le C_0, \quad |\operatorname{Re}\lambda| \to \infty. \tag{5}$$

Another way is to note that if $\lambda = is$, $s^2 > -\inf V$, then an integration by parts argument shows that there is no nontrivial solution u to the equation $(P_V - \lambda^2)u = 0$ with $u(x) \sim e^{\pm i\lambda x}$ for $\pm x \gg 1$, and thus $\mathbf{W}(\lambda) \neq 0$.

Now the meromorphy of $\mathbf{W}(\lambda)^{-1}$ implies the meromorphy of $R_V(x, y; \lambda)$ in λ and thus of the operator R_V.

6 (b) By (5), we see that for $|\operatorname{Im}\lambda| \le C_0$ and $|\lambda|$ large enough

$$|\mathbf{W}(\lambda)|^{-1} \le |\lambda|^{-1}.$$

In particular, λ is not a resonance. Next, we use the formula for $R_V(\lambda)$ and the asymptotics of $e_\pm(x)$ from Exercise 5 to see that for all $f \in L^1(\mathbb{R})$ and $\chi \in C_c^\infty(\mathbb{R})$

$$\begin{aligned}\|\chi R_V(\lambda)\chi f\|_{L^\infty} &\le \sup_{x,y} |\chi(x) R_V(x, y; \lambda)\chi(y)| \cdot \|f\|_{L^1} \\ &\le |\lambda|^{-1} \sup |\chi e_+| \cdot \sup |\chi e_-| \cdot \|f\|_{L^1} \\ &\le C|\lambda|^{-1}\|f\|_{L^1}\end{aligned}$$

where C depends only on V, C_0, χ.

7 (a) For each $a \in C^\infty(\mathbb{R})$ we have

$$(P_V - \lambda^2)e^{\pm i\lambda x}a(x) = i\lambda e^{\pm i\lambda x}\big(\mp 2\partial_x a(x) - i\lambda^{-1}V(x)a(x) + i\lambda^{-1}\partial_x^2 a(x)\big).$$

In order to have $(P_V - \lambda^2)e^{(N)}(x) = \mathcal{O}(|\lambda|^{-N})$, the functions $a_\pm^{(n)}$ should solve the system of transport equations

$$\begin{aligned}\mp 2\partial_x a_\pm^{(0)}(x) &= 0, \\ \mp 2\partial_x a_\pm^{(n+1)}(x) &= iV(x)a_\pm^{(n)}(x) - i\partial_x^2 a_\pm^{(n)}(x).\end{aligned}$$

These transport equations have unique solutions, given the boundary conditions $a_\pm^{(n)}(x) = \delta_{n0}$ for $\pm x \gg 1$. Moreover, $a_\pm^{(0)}(x) \equiv 1$ and $a_\pm^{(n)}$ is locally constant for large $|x|$. It is then easy to see that $(P_V - \lambda^2)e_\pm^{(N)}(x) = 0$ for large $|x|$.

For part (c) below, we also compute $a_\pm^{(1)}$. For $n = 0$ the transport equation gives

$$\mp 2\partial_x a_\pm^{(1)}(x) = iV(x).$$

Combining this with the initial condition $a_\pm^{(1)}(x) = 0$ for $\pm x \gg 1$, we get

$$a_+^{(1)}(x) = \frac{i}{2}\int_x^\infty V(s)\,ds, \quad a_-^{(1)}(x) = \frac{i}{2}\int_{-\infty}^x V(s)\,ds. \tag{6}$$

7 (b) Put

$$W_\pm^+(x) = W(e_+^{(N)}, e_\pm)(x).$$

Since locally uniformly in x, we have $e_\pm(x) = \mathcal{O}(1)$ by Exercise 5 and $(P_V - \lambda^2)e_+^{(N)}(x) = \mathcal{O}(|\lambda|^{-N})$ by Exercise 7(a), we find

$$\partial_x W_\pm^+(x) = e_\pm(x) \cdot (P_V - \lambda^2)e_+^{(N)}(x) = \mathcal{O}(|\lambda|^{-N}).$$

On the other hand, for large x we have $e_+^{(N)}(x) = e^{i\lambda x} = e_+(x)$ and thus

$$W_\pm^+(x) = W(e_+, e_\pm) \quad \text{for } x \gg 1.$$

Therefore we have locally uniformly in x,

$$W_\pm^+(x) = W(e_+, e_\pm) + \mathcal{O}(|\lambda|^{-N}).$$

Using the identity

$$\begin{pmatrix} e_+^{(N)}(x) \\ \partial_x e_+^{(N)}(x) \end{pmatrix} = \frac{1}{W(e_+, e_-)} \begin{pmatrix} -e_-(x) & e_+(x) \\ -e_-'(x) & e_+'(x) \end{pmatrix} \begin{pmatrix} W_+^+(x) \\ W_-^+(x) \end{pmatrix}$$

and the fact that $W(e_+, e_-) = -2i\lambda + \mathcal{O}(1)$ by (5), we get the needed bounds for $e_+ - e_+^{(N)}$. Similarly we obtain the bounds for $e_- - e_-^{(N)}$.

7 (c) Since $a_\pm^{(n)}(x)$ are locally constant for large x, we have for some constants $a_\pm^{(n)}(\infty), a_\pm^{(n)}(-\infty)$

$$a_\pm^{(n)}(x) = \begin{cases} a_\pm^{(n)}(\infty), & x \gg 1; \\ a_\pm^{(n)}(-\infty), & -x \gg 1. \end{cases}$$

Note that

$$a_\pm^{(n)}(\pm\infty) = \delta_{n0}, \quad a_\pm^{(0)}(\mp\infty) = 1, \quad a_\pm^{(1)}(\mp\infty) = \frac{i}{2}\int_{\mathbb{R}} V(s)\,ds \tag{7}$$

where the latter equation follows from (6). By Exercise 7(b) we have locally uniformly in x,

$$e_\pm(x) = \begin{cases} e^{\pm i\lambda x}, & \pm x \gg 1; \\ e^{\pm i\lambda x}\sum_{n=0}^N \lambda^{-n} a_\pm^{(n)}(\mp\infty) + \mathcal{O}(|\lambda|^{-N-1}), & \mp x \gg 1. \end{cases}$$

Recall that the scattering matrix is given by

$$S(\lambda) = \begin{pmatrix} T(\lambda) & R_+(\lambda) \\ R_-(\lambda) & T(\lambda) \end{pmatrix}$$

and $T(\lambda), R_\pm(\lambda)$ are determined as follows: for any solution u to the equation $(P_V - \lambda^2)u = 0$, u has the form

$$u(x) = \begin{cases} b_+ e^{-i\lambda x} + a_+ e^{i\lambda x}, & x \gg 1; \\ b_- e^{i\lambda x} + a_- e^{-i\lambda x}, & -x \gg 1 \end{cases}$$

and

$$\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = S(\lambda) \begin{pmatrix} b_- \\ b_+ \end{pmatrix}.$$

Applying this to $u = e_+$ we get as $|\lambda| \to \infty$

$$a_+ = 1, \quad b_+ = 0, \quad a_- = \mathcal{O}(|\lambda|^{-\infty}), \quad b_- \sim \sum_{n=0}^{\infty} \lambda^{-n} a_+^{(n)}(-\infty).$$

Similarly putting $u = e_-$ gives

$$a_- = 1, \quad b_- = 0, \quad a_+ = \mathcal{O}(|\lambda|^{-\infty}), \quad b_+ \sim \sum_{n=0}^{\infty} \lambda^{-n} a_-^{(n)}(\infty).$$

This gives the asymptotics

$$T(\lambda)^{-1} \sim \sum_{n=0}^{\infty} \lambda^{-n} a_+^{(n)}(-\infty) \sim \sum_{n=0}^{\infty} \lambda^{-n} a_-^{(n)}(\infty), \quad R_\pm(\lambda) = \mathcal{O}(|\lambda|^{-\infty}).$$

In particular, by (7) we have

$$T(\lambda)^{-1} = 1 + \frac{i}{2\lambda} \int_{\mathbb{R}} V(s)\, ds + \mathcal{O}(|\lambda|^{-2})$$

and thus

$$T(\lambda) = 1 - \frac{i}{2\lambda} \int_{\mathbb{R}} V(s)\, ds + \mathcal{O}(|\lambda|^{-2}).$$

An corollary of this asymptotic expansion is that the integral of V is determined by the scattering matrix $S(\lambda)$.