More on resolvent in the upper half-plane

$$P_V = -\partial_x^2 + V(x), \quad V \in C_c^\infty(\mathbb{R};\mathbb{R})$$

$\begin{cases} (P_V - \lambda^2)u = f \\ u \text{ outgoing} \end{cases}$ ~~has~~ has unique solution $u = R_V(\lambda) f \in C^\infty$ for all $f \in C_c^\infty$ if λ not a resonance.

Would like a __functional analytic setup__:

$$P_V - \lambda^2 : (\text{functional space}) \longrightarrow (\text{another functional space})$$

& $R_V(\lambda)$ is its inverse.

What spaces could we take?

__Option 1.__ $P_V - \lambda^2 : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$

Not invertible (2D kernel)

__Option 2.__ $P_V - \lambda^2 : C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$

also not invertible: $\exists f \in C_c^\infty$ s.t. the equation $(P_V - \lambda^2)u = f$ has no solutions $u \in C_c^\infty$

Neither option used that u should be outgoing.

But when $\operatorname{Im}\lambda > 0$, we know that outgoing solutions are exactly those which are in L^2.

__Option 3.__ (works only for $\operatorname{Im}\lambda > 0$!)

Consider $L^2(\mathbb{R})$, and the __Sobolev space__

$$H^2(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : u \text{ has weak derivatives } u', u'' \in L^2(\mathbb{R})\} \subset L^2(\mathbb{R})$$

~~Then P_V~~ Norm:

$$\|u\|^2_{H^2(\mathbb{R})} = \int_{\mathbb{R}} |u|^2 + \int_{\mathbb{R}} |u'|^2 + \int_{\mathbb{R}} |u''|^2.$$

L^2, H^2 are __Hilbert spaces__

Now, $P_V : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ bdd linear operator

$P_V - \lambda^2 : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ as well

<u>Thm 1</u> If λ, $\operatorname{Im}\lambda > 0$, is <u>not</u> a resonance, then $P_V - \lambda^2 : H^2 \to L^2$ is invertible

and $R_V(\lambda) \underset{\uparrow}{=} (P_V - \lambda^2)^{-1}$.

meaning: for $f \in C_c^\infty$, $R_V(\lambda) f = (P_V - \lambda^2)^{-1} f$.

we will ~~the~~ typically talk about P_V mapping Sobolev spaces rather than C^∞ / C_c^∞ from now on.

<u>Proof.</u> See pset 2...

Basic idea of the above proof: <u>Schur's inequality</u> can be anything!

<u>Lemma</u> Assume that an operator $T : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is defined by $Tf(x) = \int_{\mathbb{R}} K(x,y) f(y)\, dy$, $f \in L^\infty(\mathbb{R})$ where the following are finite

$$C_1 := \sup_x \int_{\mathbb{R}} |K(x,y)|\, dy, \quad C_2 := \sup_y \int_{\mathbb{R}} |K(x,y)|\, dx.$$

Then T extends to a bdd operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ (think of action of T on C_c^∞, dense in L^∞ & in L^2)

and $\|T\|_{L^2 \to L^2} \le \sqrt{C_1 \cdot C_2}$.

<u>Proof</u> Enough to show that for $f \in C_c^\infty$,

$\|Tf\|_{L^2} \le \sqrt{C_1 C_2} \cdot \|f\|_{L^2}$. We have for all x,

Hölder: $|K| = |K|^{1/2} \cdot |K|^{1/2}$

$$|Tf(x)| \le \int_{\mathbb{R}} |K(x,y)| \cdot |f(y)|\, dy \le \sqrt{\int_{\mathbb{R}} |K(x,y)|\, dy} \cdot \sqrt{\int_{\mathbb{R}} |K(x,y)| \cdot |f(y)|^2 dy}$$

$$\le \sqrt{C_1} \cdot \sqrt{\int_{\mathbb{R}} |K(x,y)| \cdot |f(y)|^2 dy}.$$

Now $\|Tf\|^2_{L^2} = \int_{\mathbb{R}} |Tf(x)|^2 dx$

$\leq C_1 \int_{\mathbb{R}^2} |K(x,y)| \cdot |f(y)|^2 dy\, dx \leq C_1 C_2 \|f\|^2_{L^2}$. $\square$

Now apply Lemma above to the formula

$$R_V(\lambda) f(x) = \int_{\mathbb{R}} R_V(x,y;\lambda) f(y) dy,$$

$$R_V(x,y;\lambda) = \frac{1}{W(\lambda)} \cdot \begin{cases} e_+(x) \cdot e_-(y), & x > y \\ e_-(x) \cdot e_+(y), & x < y \end{cases}$$

and use that $e_\pm$ are exponentially decaying when $\pm x \gg 1$....
(details in pset) to see $R_V(\lambda) : L^2 \to L^2$.

What if λ is a resonance?
Recall the following
<u>Definition</u> Assume $\mathcal{H}_1, \mathcal{H}_2$ are two Hilbert spaces.
An operator $T: \mathcal{H}_1 \to \mathcal{H}_2$ is <u>Fredholm</u>, if
1) Ker T has finite dimension, and
2) Range $T = T(\mathcal{H}_1)$ has finite codimension
(1 + 2 $\Rightarrow$ Range $T \subset \mathcal{H}_2$ is closed)
Index of T = dim Ker T − codim Range T.

<u>Thm 2</u> For any λ, $\operatorname{Im} \lambda > 0$,

$P_V - \lambda^2 : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is Fredholm of index 0.

<u>Proof</u> Pset again... $\square$

Moral of the story:

Fredholm operators are as good as matrices... so having a Fredholm problem for $R_V(\lambda)$ makes it possible to use linear algebra to understand resonant states, behavior of $R_V(\lambda)$ near a pole etc.

What about more general λ?

Complex scaling

$P_V = -\partial_x^2 + V(x)$. Want to make x a complex number.

Basic idea: an outgoing solution has the form $\sim e^{i\lambda x}$ for $\cancel{x \gg 1}$, $x \geq r_0$, $\operatorname{supp} V \subset [-r_0, r_0]$

For $\operatorname{Im} \lambda \leq 0$, this is not in L^2 on $\mathbb{R}$.

However, $e^{i\lambda z}$ is a holomorphic function of $z \in \mathbb{C}$.

Let's fix θ, $0 < \theta < \pi/2$ (scaling angle; can actually take $0 < \theta < \pi$ but harder to draw).

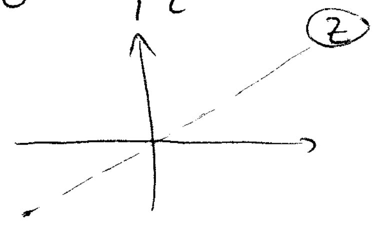
And put $z := e^{i\theta} \cdot s$, $s > r_0$.

Then $e^{i\lambda z} = e^{i\lambda e^{i\theta} \cdot s}$ is exponentially decaying as $s \to \infty$ as long as $\operatorname{Re}(i\lambda e^{i\theta}) < 0$

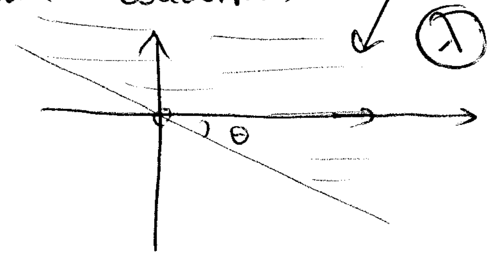
that is, $\operatorname{Im}(\lambda e^{i\theta}) > 0$ or $-\theta < \arg \lambda < \pi - \theta$

Similarly $e^{-i\lambda z}$ is exponentially decaying as $s \to -\infty$ on $\{z = e^{i\theta} \cdot s\}$ under the same conditions

z

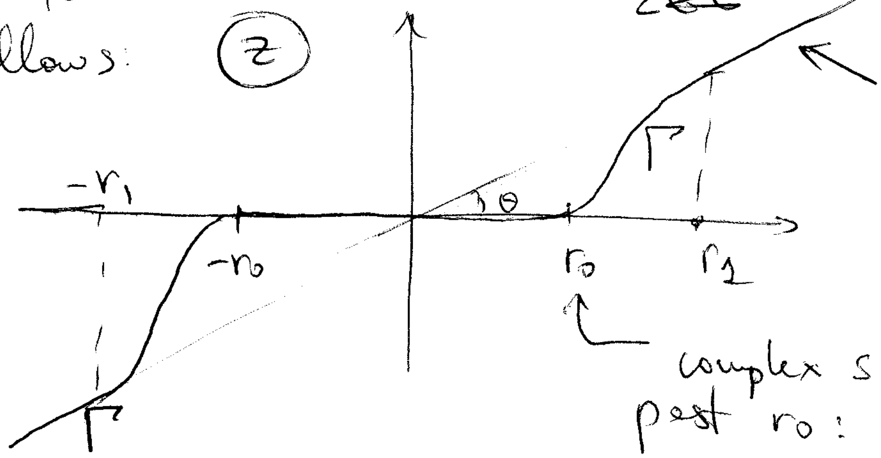


λ



Idea of complex scaling:

~~const~~ deform P_V to an operator $P_{V,\Gamma}$ ~~on the follo~~ on a contour $\Gamma \subset \mathbb{C}$ ~~$z \in \mathbb{C}$~~ which looks as follows:



for $|z| > r_1$, Γ coincides with $e^{i\theta}\cdot\mathbb{R}$

complex scaling starts past r_0: $[-r_0, r_0] \subset \Gamma$

How ~~do~~ define $P_{V,\Gamma}$?

Parametrize Γ by a real parameter s, say $s = \operatorname{Re} z$.

So $\Gamma = \{ z = \gamma(s),\ s \in \mathbb{R} \}$, $\gamma : \mathbb{R} \to \mathbb{C}$.

Take $f \in C^\infty(\Gamma)$ ~~$\to$ can view $f = f$~~

have a fn. $f \circ \gamma \in C^\infty(\mathbb{R})$.

Apply chain rule formally:

$$\cancel{\partial_x}\ \partial_s (f \circ \gamma)(s) = \cancel{\partial_x f}\ \partial_z f(\gamma(s)) \cdot \gamma'(s).$$

So, define ~~for~~ $f \in C^\infty(\Gamma)$,

$\partial_z^\Gamma f \in C^\infty(\Gamma)$ by

$$\partial_z^\Gamma f(\gamma(s)) = \gamma'(s)^{-1} \cdot \partial_s (f \circ \gamma(s)).$$

Does not depend on parametrization

Put $P_{V,\Gamma} = -(\partial_z^\Gamma)^2 + V$, where

$$Vf(z) = \begin{cases} V(z) f(z), & z \in \mathbb{R} \\ 0, & \text{otherwise} \end{cases} \to \text{makes sense as } \operatorname{supp} V \subset [-r_0, r_0] \subset \Gamma \cap \mathbb{R}.$$

We see that $P_{V,\Gamma}$ is still a 2^nd^ order differential operator:

In a parametrization $z = \gamma(s)$,

$$P_{V,\Gamma} = -\left(\frac{1}{\gamma'(s)}\partial_s\right)^2 + V(s).$$

In particular, if $\gamma(s) = e^{i\theta}s$, $|s| \geq r_1$, then for $|s| \geq r_1$ we compute

$$P_{V,\Gamma} = -e^{-2i\theta}\partial_s^2.$$

Solutions to $(P_{V,\Gamma}-\lambda^2)u=0$ ~~st~~ have the form

$$\exp(\pm i\lambda e^{i\theta}\cdot s).$$

In particular, if $-\theta < \arg\lambda < \pi-\theta$ then

$\exp(\pm i\lambda e^{i\theta}\cdot s)$ is exponentially decaying

as $s \to \pm\infty$. Using same argument as for P_V in $\operatorname{Im}\lambda > 0$ (see pset), we obtain

<u>Thm 3.</u> For $-\theta < \arg\lambda < \pi - \theta$,
(use s-parametrization, $\sim H^2(\mathbb{R})$)

$P_{V,\Gamma}-\lambda^2 : H^2(\Gamma) \to L^2(\Gamma)$ is Fredholm of index 0.

It has a meromorphic inverse

$$R_{V,\Gamma}(\lambda) = (P_{V,\Gamma}-\lambda^2)^{-1} : L^2(\Gamma) \to H^2(\Gamma),$$
$$-\theta < \arg\lambda < \pi-\theta.$$

How are $R_V(\lambda)$ and $R_{V,\Gamma}(\lambda)$ related?

<u>Thm 4</u> Assume $f \in C_c^\infty(\mathbb{R})$ and $\chi \in C_c^\infty(\mathbb{R})$

supp $\chi \subset [-r_0, r_0] \subset \Gamma \cap \mathbb{R}$.

Then $\chi R_V(\lambda)\chi f = \chi R_{V,\Gamma}(\lambda)\chi f$

when $-\theta < \arg\lambda < \pi - \theta$.

<u>Remark</u>: we have now defined resonances in $-\theta < \arg\lambda < \pi - \theta$ as eigenvalues of a Fredholm problem (for $P_{V,\Gamma}$).

<u>Proof</u>. Enough to show this when λ not a pole of $R_V(\lambda)$ or $R_{V,\Gamma}(\lambda)$ (unique continuation in λ)

Put $u := R_V(\lambda)\chi f \in C^\infty(\mathbb{R})$. Extend it holomorphically to $u^{\mathbb{C}}$ on $\mathbb{R} \cup \{|\mathrm{Re}\, z| > r_0\}$.

Can do it since $u \sim e^{\pm i\lambda x}$ for $\mathrm{Re}|x| > r_0$.

Now, let $u^\Gamma := u^{\mathbb{C}}|_\Gamma \in C^\infty(\Gamma)$.

Then
- $(P_{V,\Gamma} - \lambda^2)u^\Gamma = \chi f$
- $u^\Gamma \in H^2(\Gamma)$ due to exponential decay at infinity

Thus $u^\Gamma = R_{V,\Gamma}(\lambda)\chi f$.

Now $\chi R_{V,\Gamma}(\lambda)\chi f = \chi u^\Gamma = \chi u = \chi R_V(\lambda)\chi f$. $\square$