Recall:

For $a \in S^k_{1,0,h}$, can define $Op_h(a): \mathcal{S} \to \mathcal{S}$, $\mathcal{S}' \to \mathcal{S}'$,
$H^s_h \to H^{s-k}_h$.

We previously introduced the spaces of classical symbols.

Hom^k – positively homogeneous of order k

$S^k \subset S^k_{1,0}$: symbols that are $\underset{|\xi|\to\infty}{\sim} \sum_{j=0}^{\infty} a_j$, $a_j \in Hom^{k-j}$

$S^k_h \subset S^k_{1,0,h}$: symbols that are $\overset{h\to 0}{\underset{|\xi|\to\infty}{\sim}} \sum_{\ell} h^{\ell} a_\ell$, $a_\ell \in S^{k-\ell}$

Def. We say that $A: \mathcal{S}' \to \mathcal{S}'$ is in $\Psi^k_h(\mathbb{R}^n)$,
if $A = Op_h(a)$ for some $a \in S^k_h$.

We say that $A = O(h^\infty)_{\Psi^{-\infty}}$ if ~~$A \in h^N \Psi^{-N}_h$~~
$A = Op_h(a)$ for some $a \in h^\infty S^{-\infty}$.

Note: $A = O(h^\infty)_{\Psi^{-\infty}} \Rightarrow \|A\|_{H^{-N}_h \to H^N_h} \le C_N h^N \ \forall N$.

~~In general we would want to can~~

Principal symbol

For $A = Op_h(a) \in \Psi^k_h(\mathbb{R}^n)$, define the semiclassical principal symbol
$\sigma_h(A) \in S^k$ by $\sigma_h(A) := a_0$ where
$a \sim \sum_\ell h^\ell a_\ell$, $a_\ell \in S^{k-\ell}$. That is, $a_0 \in S^k$ and
$a = a_0 + h S^{k-1}_h$.

Based on what we proved last time, we get the following properties:

1. $\sigma_h : \Psi_h^k(\mathbb{R}^n) \to S^k(\mathbb{R}^{2n})$ is <u>onto</u>.

   (indeed, $\sigma_h(\mathrm{Op}_h(a)) = a$, $a \in S^k(\mathbb{R}^{2n})$)

2. For $A \in \Psi_h^k(\mathbb{R}^n)$, $\sigma_h(A) = 0 \iff A \in h\Psi_h^{k-1}(\mathbb{R}^n)$

   (immediate from the definition of σ_h)

3. For $A \in \Psi_h^k(\mathbb{R}^n)$, $B \in \Psi_h^\ell(\mathbb{R}^n)$, $AB \in \Psi_h^{k+\ell}(\mathbb{R}^n)$

   and 3a $\sigma_h(AB) = \sigma_h(A)\,\sigma_h(B)$

   3b $\sigma_h\left(\frac{i}{h}[A,B]\right) = \{\sigma_h(A), \sigma_h(B)\}$.

4. For $A \in \Psi_h^k(\mathbb{R}^n)$, $A^* \in \Psi_h^k(\mathbb{R}^n)$ and

   $\sigma_h(A^*) = \overline{\sigma_h(A)}$.

<u>Note</u>: 1 + 2 give a <u>short exact sequence</u>

$$\begin{tikzcd} 0 \arrow[r] & h\Psi_h^{k-1} \arrow[r] & \Psi_h^k \arrow[r, "\sigma_h"] & S^k \arrow[r] \arrow[l, bend left, dashed, "\mathrm{Op}_h"] & 0 \end{tikzcd}$$

READ [Dy Zw, §§ E.1.5, E.1.6]

<u>Operators on manifolds</u>

If M is a manifold, then we can use coordinate charts to still define the class $\Psi_h^k(M)$ & 1 - 4 still hold, but:

- $S^k(\mathbb{R}^{2n})$ is replaced by $S^k(T^*M)$,
  $T^*M \to$ cotangent bundle : $T^*M = \{(x,\xi) \mid x \in M,\ \xi \in T_x^*M\}$
- If M noncompact – usually just require estimates on compact sets, so $\Psi_h^k : H^s_{h,\mathrm{comp}} \to H^{s-k}_{h,\mathrm{comp}}$,
  $H^s_{h,\mathrm{loc}} \to H^{s-k}_{h,\mathrm{loc}}$
- Op_h is not canonical – many choices!

Elliptic set & WF set of operators

We will still work on $\mathbb{R}^n$ but the statements & the proofs do in fact apply to any manifold.

For $A \in \Psi_h^k(\mathbb{R}^n)$, we want to associate to it 2 sets:

- Wavefront set $WF_h(A)$: "the support of the full symbol of A, modulo h^∞"
- Elliptic set $\mathrm{ell}_h(A)$: "the set where ~~the~~ $\sigma_h(A)$ does not vanish"

But we have to be careful with $|\xi| \to \infty$.

For instance, $a(x,\xi;h) = e^{-1/h}$ is $O(h^\infty)$ everywhere but it is not in $h^\infty S^{-\infty}$ & $Op_h(a)$ does not map $H_h^{-N} \to H_h^N \ldots$

A neat way to deal with this is the $\boxed{\text{READ } [\mathrm{Dy}\,\mathrm{Zw},\ \S E.1.2\ldots]}$

<u>Fiber-radially compactified cotangent bundle</u> $\overline{T^*\mathbb{R}^n}$.

We have $T^*\mathbb{R}^n = \mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ ~~For each x~~,

~~we map~~ Define $\overline{T^*\mathbb{R}^n} = \mathbb{R}^n_x \times \overline{\mathbb{R}^n_\xi}$ where $\overline{\mathbb{R}^n_\xi} \simeq \overline{B(0,1)} \subset \mathbb{R}^n$ and $\mathbb{R}^n \hookrightarrow \overline{\mathbb{R}^n}$ by the map

$$\xi \in \mathbb{R}^n \longmapsto \eta = \frac{\xi}{1+\langle\xi\rangle} \in B(0,1).$$

So, the interior of the ball is $\mathbb{R}^n$ & the bdary $\partial\overline{\mathbb{R}^n} \simeq \mathbb{S}^{n-1}$ ~~is~~ corresponds to ξ going to infinity in different directions.

<u>Note</u>: ~~the~~ a defining fn. of the bdary is $1-|\eta|^2 =$

$$= 1 - \frac{|\xi|^2}{(1+\langle\xi\rangle)^2} = \frac{1+\langle\xi\rangle^2+2\langle\xi\rangle-|\xi|^2}{(1+\langle\xi\rangle)^2} \sim |\xi|^{-1} \text{ as } |\xi| \to \infty.$$

Now can define WF_h & ell_h! $\boxed{\text{READ [DyZw, §§E.2.1, E.2.2]}}$

Def. Let $A \in \Psi_h^k(\mathbb{R}^n)$. We say that $(x_0, \xi_0) \in \overline{T^*\mathbb{R}^n}$ does NOT lie in $WF_h(A)$, if $A = Op_h(a)$ & $\exists$ a nbhd U of (x_0, ξ_0) in $\overline{T^*\mathbb{R}^n}$ such that $\partial_x^\alpha \partial_\xi^\beta a = O(h^\infty \langle\xi\rangle^{-\infty})$ ~~with~~ in U.
This defines $WF_h(A) \subset \overline{T^*\mathbb{R}^n}$ closed set.

Note: if $A = e^{-1/h} \cdot Id$, $a = e^{-1/h}$, then
$$WF_h(A) = \partial \overline{T^*\mathbb{R}^n} = \text{"}\{(x,\xi) \mid |\xi| = \infty\}\text{"}.$$

Principal symbol: If $A \in \Psi_h^k(\mathbb{R}^n)$, then $\sigma_h(A) \in S^k$.
Then it is immediate to check that $\langle\xi\rangle^{-k}\sigma_h(A)$ extends to a C^∞ function on $\overline{T^*\mathbb{R}^n}$.

Def. Let $A \in \Psi_h^k(\mathbb{R}^n)$. The elliptic set $\mathrm{ell}_h(A) \subset \overline{T^*\mathbb{R}^n}$ is the set ~~where~~
$$\mathrm{ell}_h(A) = \{ \cancel{\langle\xi\rangle^{-k}} (x,\xi) \mid \langle\xi\rangle^{-k}\sigma_h(A)(x,\xi) \neq 0 \}.$$

Note: $\mathrm{ell}_h(A) \subset \overline{T^*\mathbb{R}^n}$ is open. For $\xi \in T^*\mathbb{R}^n$, just need $\sigma_h(A)(x,\xi) \neq 0$. For $(x,\eta), \eta \in \partial \overline{T_x^*\mathbb{R}^n} \sim S^{n-1}$, need $|\sigma_h(A)(x,\xi)| > c|\xi|^k$ for $|\xi| \gg 1$, $\frac{\xi}{|\xi|} \approx \eta$.

One more definition: if $A = A(h) : C_c^\infty(U) \to \mathcal{D}'(U)$, $U \subset \mathbb{R}^n$, then A is compactly supported on U if $\exists\, \chi \in C_c^\infty(U)$ h-indepdt such that $A = \chi A \chi$ for all h.

Note: $A \in \Psi_h^k(\mathbb{R}^n)$ is compactly supported $\Rightarrow$ $WF_h(A) \subset \overline{T^*\mathbb{R}^n}$ is compact (i.e. compact in x.)

Elliptic parametrix

Let $P \in \Psi_h^k(\mathbb{R}^n)$. We want to understand solutions to $Pu = f$: $\|u\|_? \lesssim C\|f\|_? +$ remainder.
However it is often useful to estimate u using different methods in different parts of phase space $T^*\mathbb{R}^n$.
So we'd like to first do $\|Au\|_? \lesssim C\|f\|_? +$ remainder.
The easiest situation is when we can write $\boxed{A = QP + \text{rem.}}$ for some Q. It is given by

Thm. [[DyZw, Proposition E.31] Assume $P \in \Psi_h^k(\mathbb{R}^n)$, $A \in \Psi_h^0(\mathbb{R}^n)$, A is compactly supported, and

$$\boxed{WF_h(A) \subset \text{ell}_h(P).}$$

Then there exists $Q \in \Psi_h^{-k}(\mathbb{R}^n)$ such that

$$\boxed{A = QP + O(h^\infty)_{\Psi^{-\infty}}.}$$

Proof ① First construct $Q_0 \in \Psi_h^{-k}(\mathbb{R}^n)$ s.t.
$A = Q_0 P + h\Psi_h^{-1}(\mathbb{R}^n)$. For that, note that by the assumption $WF_h(A) \subset \text{ell}_h(P)$, we get $\text{supp}\, \sigma_h(A) \subset \text{ell}_h(P)$ & thus, denoting $a_0 := \sigma_h(A)$, $p := \sigma_h(P)$, we have $|p(x,\xi)| \geq c\langle\xi\rangle^k$ near $\text{supp}\, a$.
Then we can define $q_0 := a_0/p$ and verify directly that

$$q_0 \in S_h^{-k}.$$

Put $Q_0 := Op_h(q_0) \in \Psi_h^{-k}(\mathbb{R}^n)$. Then by the Product Rule,

$A - Q_0 P \in \Psi_h^0(\mathbb{R}^n)$ and

$\sigma_h(A - Q_0 P) = a_0 - q_0 p = 0$. So

$A - Q_0 P \in h\Psi_h^{-1}(\mathbb{R}^n)$.

2 Now we iterate. Note that $WF_h(Q_0) \subset \cancel{\operatorname{supp} q_0}\ WF_h(A)$ by construction of q_0. So we have

$A = Q_0 P + hR_0, \quad R_0 \in \Psi_h^{-1}(\mathbb{R}^n), \; WF_h(R_0) \subset WF_h(A)$.

Repeating the process 1 for $-R_0$ in place of A we get

$Q_1 \in \Psi_h^{-k-1}(\mathbb{R}^n): \; -R_0 = Q_1 P + h\Psi_h^{-2}(\mathbb{R}^n)$.

Thus $A = (Q_0 + hQ_1)P + h^2 R_1, \; R_1 \in \Psi_h^{-2}(\mathbb{R}^n) \ldots$

~~Write~~ ~~Put~~ ~~q~~ ~~$Q :=$~~ Put $Q := Op_h(q), \; q \sim \sum_{j=0}^{\infty} h^j q_j$,

$Q_j = Op_h(q_j)$ constructed iteratively, $Q_j \in \Psi_h^{-k-j}$

Then $Q - \sum_{j=0}^{J-1} h^j Q_j \in h^J \Psi_h^{-k-J}$. So

And $A = \left(\sum_{j=0}^{J-1} h^j Q_j\right) P + h^J \Psi_h^{-J}$.

$A = QP + h^J \Psi_h^{-J} \quad \forall J \Rightarrow A = QP + O(h^\infty)_{\Psi^{-\infty}}$

as needed. $\square$

We can now prove the Elliptic Estimate:

Thm [DyZw, Theorem E.32] Assume that

- $P \in \Psi_h^k(\mathbb{R}^n)$ is a semiclassical differential ~~operator~~ (only need it to be differential to be able to apply it to fns. defined on $U \subset \mathbb{R}^n$. Properly supported on U would do.)
- $A \in \Psi_h^0(\mathbb{R}^n)$ is compactly supported on some open set $U \subset \mathbb{R}^n$, i.e. $\exists \chi_A \in C_c^\infty(U)$ s.t. $A = \chi_A A \chi_A$.
- $WF_h(A) \subset \mathrm{ell}_h(P)$

Then $\exists \chi \in C_c^\infty(U)$ s.t. the following holds.
Let $u \in \mathcal{D}'(U)$ and ~~assume that~~ put $f := Pu \in \mathcal{D}'(U)$.
Assume that for some s, $\chi f \in H^s$ (note: $\chi f \in \mathcal{E}'(\mathbb{R}^n)$)
Then $Au \in H^{s+k}$ and $\forall N$, $\forall h \in (0,1]$

$$\|Au\|_{H_h^{s+k}} \leq C\|\chi f\|_{H_h^s} + \cancel{O(h^\infty)} C_N \|\chi u\|_{H_h^{-N}}$$

where C, C_N do not depend on h or u.
(they depend on P, A, U, s).

Proof. Mostly just need to fight with cutoffs.
Denote $\chi_1 \prec \chi_2$ if ~~$\chi_2 = 1$ near supp~~ $\mathrm{supp}\,\chi_1 \cap \mathrm{supp}(1-\chi_2) = \emptyset$,
we ~~set~~ fix $\chi_1, \chi \in C_c^\infty(U)$ s.t.

$$\chi_A \prec \chi_1 \prec \chi.$$

Put $v := \chi_1 u \in \mathcal{E}'(U) \subset \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$.
Then $Au = \chi_A A \chi_A u = \chi_A A v$. Let Q be the elliptic parametrix, i.e. $Q \in \Psi_h^{-k}$, $A = QP + \underbrace{O(h^\infty)_{\Psi^{-\infty}}}_{\text{call this } R}$.
Then $Au = \chi_A A v = \chi_A Q P v + \chi_A R v$.

Again:

$$Au = \chi_A Q P v + \chi_A R \chi_1 u.$$

We show $Au \in H^{s+k}$ & estimate $\|Au\|_{H_h^{s+k}}$:

- $\chi_A R \chi_1 = O(h^\infty)_{\cancel{\;}\mathcal{D}'(U) \to C_c^\infty(U)}$

and $\chi_1 = \chi_1 \chi$, so $\forall N$,

$$\|\chi_A R \chi_1 u\|_{H_h^{s+k}} \leq C_N h^N \|\chi u\|_{H_h^{-N}}.$$

(note: $\chi u \in \mathcal{E}'(\cancel{\;}\mathbb{R}^n) \Rightarrow \chi u \in H_h^{-N}$ for some N.)

- We write $Pv = P\chi_1 u = \chi_1 f + [P, \chi_1] u$.

$\|\chi_A Q \chi_1 f\| \leq C \|\chi f\|_{H_h^s}$ since $Q \in \Psi_h^{-k}$.

- Finally, we need to look at

$$\chi_A Q [P, \chi_1] u = \chi_A Q [P, \chi_1] \chi u.$$

However, $[P, \chi_1]$ is a diff. operator supported away from supp χ_A. By pseudolocality of Q, we get

~~$\chi_A Q [P, \chi_1] \in h^\infty$~~ $\chi_A Q [P, \chi_1] = O(h^\infty)_{\Psi^{-\infty}}$,

so $\|\chi_A Q [P, \chi_1] u\|_{H_h^{s+k}} \leq C_N h^N \|\chi u\|_{H_h^{-N}}$ $\forall N$.

$\square$

Note: pseudolocality of Q was crucial in the last step.

The wave operator also has a parametrix, but it is not pseudolocal, so we ~~do not~~ might have nonsmooth solns with C^∞ right-hand side.

A nonsemiclassical application:

Thm. [Elliptic regularity]

Assume $P = \sum_{|\alpha| \le k} a_\alpha(x) D_x^\alpha$, $a_\alpha \in C^\infty(U)$,

and P is (nonsemiclassically) elliptic, i.e. $\exists c > 0$:

$$\Big| \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha \Big| \ge c |\xi|^k \text{ for all } x \in U, |\xi| > 1.$$

Then $\forall u \in \mathcal{D}'(U)$, if $Pu \in C^\infty(U)$, then $u \in C^\infty(U)$.

Proof Define $\hat{P} := h^k P$. (can fix $h := 1$ or take any h, makes no difference!)

The statement is local, so we can shrink U a bit & extend the coefficients of P to $\mathbb{R}^n$, so that $\hat{P} \in \Psi_h^k(\mathbb{R}^n)$. Now,

$\sigma_h(\hat{P})(x,\xi) = \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha$ since the $|\alpha| < k$ terms are $O(h)$.

So, $\mathrm{ell}_h(\hat{P}) \supset \{(x,\xi) \in \overline{T^*\mathbb{R}^n} : x \in U, \xi \ne 0\}$.

To deal with 0 section, fix $\chi_1 \in C_c^\infty(U)$

$b \in C_c^\infty(\mathbb{R}^{2n})$, $b = 1$ near $\{(x,0) \mid x \in \operatorname{supp} \chi_1\}$.

Put $B := \mathrm{Op}_h(b)\chi_1 \in \Psi_h^{-N}(\mathbb{R}^n)$ $\forall N$.

Enough to show that for any $\chi_0 \in C_c^\infty(U)$, we have $\chi_0 u \in C^\infty$ (note: $\chi_0 u \in \mathcal{E}'(U) \subset \mathcal{E}'(\mathbb{R}^n)$).

Write $\chi_0 u = \chi_0 B u + \chi_0 (I - B) u$.

Take $\chi_1 \succ \chi_0$. Then:

• $\chi_0 Bu \in C^\infty$ since

$\chi_0 Bu = \chi_0 Op_h(b)\chi_1 u$ and

$\chi_0 Op_h(b)\chi_1 : \mathcal{D}' \to C^\infty$ since b is compactly supported in x, ξ.

• For $\chi_0(I-B)u$:

$\chi_0(I-B) \cancel{\mathcal{E}} = \chi_0 - \chi_0 Op_h(b)\chi_1 \in \Psi_h^0(\mathbb{R}^n)$

is compactly supported in U and

$WF_h(\chi_0(1-B)) \underset{\substack{(\text{since} \\ \chi_0 \prec \chi_1)}}{=} WF_h(\chi_0 Op_h(1-b))$, so

$WF_h(\chi_0(1-B)) \subset \{(x,\xi) \mid x \in U, \xi \neq 0\}$.

$WF_h(\chi_0(1-B))$ $\subset \mathrm{ell}_h(P)$

Thus $WF_h(\chi_0(1-B))$

& we apply the elliptic estimate:

$Pu \in H^s_{loc}(U)\ \forall s \implies \chi_0(I-B)u \in H^{s+k}_{loc}\ \forall s$

$\implies \chi_0(1-B)u \in C^\infty$. $\square$

<u>Easy application to scattering theory</u>:

if $P = -h^2\Delta + V$, $V \in C_c^\infty(\mathbb{R}^n)$, and

$Pu = 0$, then $u \in C^\infty$.

<u>Next week</u>: Propagation of Singularities