Scattering problem: $(\lambda \in \mathbb{R}\setminus\{0\})$

given $g \in C^\infty(\mathbb{S}^{n-1})$, find $f \in C^\infty(\mathbb{S}^{n-1})$, $v \in H^2_{loc}(\mathbb{R}^n)$:

$$(*)\begin{cases}(P_V-\lambda^2)v=0\\ v(r\theta)=r^{-\frac{n-1}{2}}\left(e^{i\lambda r}f(\theta)+e^{-i\lambda r}g(\theta)\right)+O(r^{-\frac{n+1}{2}})\end{cases}$$

(+ ∂_r-derivatives)

as $r\to\infty$, $\theta\in\mathbb{S}^{n-1}$.

Thm. For each g $\exists!$ f, v solving $(*)$

Proof. 1 Uniqueness: assume $g\equiv 0$.

Then v satisfies Sommerfeld Radiation Condition:

$$(\partial_r - i\lambda)v(x) = o(|x|^{-\frac{n-1}{2}})$$

By Rellich's Uniqueness Thm (stronger version, see Thm 3.32)

we have $v\equiv 0$.

2 The case $V\equiv 0$: if we write

$$u_0(x) := \int_{\mathbb{S}^{n-1}} g(\omega)e^{-i\lambda\langle x,\omega\rangle}dS(\omega),$$ then

$$u_0(r\theta) = (\lambda r)^{\frac{1-n}{2}}\left(c_n^+ e^{-i\lambda r}g(\theta)+c_n^- e^{i\lambda r}g(-\theta)\right) + O(r^{-\frac{n+1}{2}})$$

where $c_n^\pm = (2\pi)^{\frac{n-1}{2}}e^{\pm\frac{\pi}{4}(n-1)i}$

(did it last time using stationary phase)

3 For the general case we want to replace $e^{-i\lambda\langle x,\omega\rangle}$ by a distorted plane wave:

$w(x,\lambda,\omega)$ s.t. (double u, omega)

$$\begin{cases}(P_V-\lambda^2)w=0\\ w=e^{-i\lambda\langle x,\omega\rangle}+u(x,\lambda,\omega),\end{cases}$$ u is λ-outgoing

To construct w, we solve for u:

$$(P_V - \lambda^2)u = -(P_V-\lambda^2)e^{-i\lambda\langle x,\omega\rangle} = -Ve^{-i\lambda\langle x,\omega\rangle}$$

$\in L^2_{comp}$

since $(-\Delta-\lambda^2)e^{-i\lambda\langle x,\omega\rangle} = 0$.

So, put $w = e^{-i\lambda\langle x,\omega\rangle} + u$ where

$$\boxed{u = -R_V(\lambda)(Ve^{-i\lambda\langle x,\omega\rangle})}$$

(4) Now let's ~~write~~ put

$$v(x) := c_{n,\lambda}\int_{S^{n-1}} g(\omega)\,w(x,\lambda,\omega)\,dS(\omega)$$

for a well-chosen $c_{n,\lambda}\in\mathbb{C}$. We get:

- $(P_V-\lambda^2)v = 0$ since $(P_V-\lambda^2)w=0$.
- $v(x) = c_{n,\lambda}\int_{S^{n-1}} e^{-i\lambda\langle x,\omega\rangle} g(\omega)\,dS(\omega)$

$$+\, c_{n,\lambda}\int_{S^{n-1}} g(\omega)\,u(x,\lambda,\omega)\,dS(\omega).$$

The second term is <u>outgoing</u>. The first term is... at $x=r\theta$

$$c_{n,\lambda}\cdot\lambda^{\frac{1-n}{2}}(2\pi)^{\frac{n-1}{2}}e^{\frac{\pi}{4}(n-1)i}\,r^{\frac{1-n}{2}}e^{-i\lambda r}g(\theta)$$

$$+\,c_{n,\lambda}\lambda^{\frac{1-n}{2}}(2\pi)^{\frac{n-1}{2}}e^{-\frac{\pi}{4}(n-1)i}\,r^{\frac{1-n}{2}}e^{i\lambda r}g(-\theta)$$

$$+\,O(r^{-\frac{n+1}{2}}).$$

So we put $c_{n,\lambda} := \left(\frac{\lambda}{2\pi}\right)^{\frac{n-1}{2}}e^{-\frac{\pi}{4}(n-1)i}$.

Then v has the correct asymptotic behavior.

Scattering operator

In the context of the scattering problem (*), define the absolute scattering operator $S_{abs}(\lambda): C^\infty(\mathbb{S}^{n-1}) \circlearrowleft$ by

$$S_{abs}(\lambda): g \longmapsto f.$$

What is this operator for $V \equiv 0$? We then have $u \equiv 0$, $w = e^{-i\lambda\langle x, \omega\rangle}$, so from Step 4 in the previous Thm,

$$v(r\theta) = r^{\frac{1-n}{2}} e^{-i\lambda r} g(\theta) + i^{1-n} r^{\frac{1-n}{2}} e^{i\lambda r} g(-\theta) + O(r^{-\frac{n+1}{2}})$$

so $S_{abs}(\lambda) g(\theta) = i^{1-n} g(-\theta)$. for $V \equiv 0$.

It is more appealing if the scattering ~~matrix~~ operator is the identity when $V \equiv 0$. So, define in general

$$S(\lambda): C^\infty(\mathbb{S}^{n-1}) \circlearrowleft,$$

$$S(\lambda) g(\theta) = i^{n-1} S_{abs}(\lambda)(g(-\cdot))(\theta),$$

i.e. $S(\lambda): i^{1-n} g(-\theta) \mapsto f(\theta)$.

How to express it? Recall plane waves $w = e^{-i\lambda\langle x,\omega\rangle} + u$.

u outgoing $\Rightarrow$ ~~$u(x, \lambda, \omega)$~~

$$u(r\theta, \lambda, \omega) = (2\pi)^{\frac{n-1}{2}} e^{-\frac{\pi}{4}(n-1)i} (\lambda r)^{-\frac{n-1}{2}} e^{i\lambda r} b(\lambda, \theta, \omega) + O(r^{-\frac{n+1}{2}})$$

for some function $b(\lambda, \theta, \omega)$. Then from Step 4 above,

$$v(r\theta) = r^{\frac{1-n}{2}} e^{-i\lambda r} g(\theta) + i^{1-n} e^{i\lambda r} g(-\theta) + i^{1-n} \int_{\mathbb{S}^{n-1}} r^{\frac{1-n}{2}} e^{i\lambda r} b(\lambda, \theta, \omega) g(\omega)\, dS(\omega). + O(r^{-\frac{n+1}{2}})$$

So, $f(\theta) = i^{1-n} g(-\theta) + i^{1-n} \int_{S^{n-1}} b(\lambda,\theta,\omega) g(\omega)\, dS(\omega)$.

Thus we get a formula for $S(\lambda)$:

$S(\lambda) = I + A(\lambda)$ where

$$A(\lambda) g(\theta) = \int_{S^{n-1}} b(\lambda, \theta, -\omega)\, g(\omega)\, dS(\omega).$$

What can we say about the <u>regularity</u> of $A(\lambda)$?

Note that $b(\lambda, \theta, \omega)$ is the coefficient in the ~~scattering~~ asymptotics of $u(x, \lambda, \omega)$.

But $u(x,\lambda,\omega) = - R_V(\lambda)\left(V e^{-i\lambda\langle \cdot, \omega\rangle}\right)(x)$

Recall the formula for $R_V(\lambda)$: get

$$u = -R_0(\lambda)\,(I + V R_0(\lambda)\rho)^{-1}\,(I - VR_0(\lambda)(1-\rho))\left(V e^{-i\lambda\langle\cdot,\omega\rangle}\right)$$

(recall here $\rho V = V$ so $(1-\rho) V e^{-i\lambda\langle\cdot,\omega\rangle} = 0$)

$$= -R_0(\lambda)\,(I + VR_0(\lambda)\rho)^{-1}\, V e^{-i\lambda\langle\cdot,\omega\rangle}$$

How to get the ~~outgoing~~ coefficient from here?

Recall: $R_0(\lambda) g(r\theta) = \frac{1}{4\pi}\left(\frac{\lambda}{2\pi i}\right)^{\frac{n-3}{2}} \hat{g}(\lambda\theta)\cdot r^{\frac{1-n}{2}} e^{-i\lambda r} + O\left(r^{-\frac{n+1}{2}}\right)$

for $g \in L^2_{comp}$.

So, $b(\lambda, \theta, \omega) =$

$$= \tilde{C}_{n,\lambda}\cdot E_\rho(\lambda)\,(I + VR_0(\lambda)\rho)^{-1}\, V e^{-i\lambda\langle\cdot,\omega\rangle}(\theta)$$

where $E_\rho(\lambda) g(\theta) = \widehat{\rho g}(\lambda\theta) = \int_{\mathbb{R}^n} e^{-i\lambda\langle\theta, x\rangle} \rho(x) g(x)\, dx$

(we can put ρ since $(1-\rho)(I + VR_0(\lambda)\rho)^{-1} V e^{-i\lambda\langle\cdot,\omega\rangle} = 0$)

(see Thm 3.38)

Again: explicit fn

$$b(\lambda,\theta,\omega) = \tilde{c}_{n,\lambda} \cdot \left(E_\rho(\lambda)(I+VR_0(\lambda)\rho)^{-1} V e^{-i\lambda\langle\cdot,\omega\rangle}\right)(\theta),$$

$\lambda \in \mathbb{R}\setminus\{0\}$, $\theta,\omega \in \mathbb{S}^{n-1}$.

Note that $E_\rho : L^2(\mathbb{R}^n) \to C^\infty(\mathbb{S}^{n-1})$.

From this formula we see that, denoting by $k(x,y;\lambda)$ the distributional integral kernel of $(I+VR_0(\lambda)\rho)^{-1}$,

$$b(\lambda,\theta,\omega) = \tilde{c}_{n,\lambda} \cdot \int_{\mathbb{R}^{2n}} e^{-i\lambda\langle\theta,x\rangle} \rho(x)\, k(x,y;\lambda) V(y) e^{-i\lambda\langle\omega,y\rangle}\, dx\,dy$$

From here we get: $b(\lambda,\theta,\omega)$ is C^∞ in θ,ω

So $A(\lambda)$ has C^∞ integral kernel

$\Updownarrow$

it is a smoothing operator.

In particular $A(\lambda): L^2(\mathbb{S}^{n-1})\circlearrowleft$ and so

$$S(\lambda) = I + A(\lambda): L^2(\mathbb{S}^{n-1})\circlearrowleft.$$

Moreover, by examining the above formula (which write $A(\lambda)$ as a product (holom. operator). $(I+VR_0(\lambda)\rho)^{-1}$. (holom. operator)), we see that $A(\lambda): L^2(\mathbb{S}^{n-1})\circlearrowleft$ is meromorphic in $\lambda\in\mathbb{C}$,

Poles of $A(\lambda)$ $\subset$ {resonances}.

Thm. [Unitarity] (Thm 3.40) We have

$$S(\lambda)^{-1} = S(\bar{\lambda})^*, \quad \lambda \in \mathbb{C}.$$

In particular, $\lambda \in \mathbb{R} \Rightarrow S(\lambda)^* = S(\lambda)^{-1}$.
This means that $S(\lambda)$ is ~~holom~~ unitary & holomorphic on $\mathbb{R}$ (including 0, which may be a resonance!)

Proof

Enough to show that $S_{abs}(\lambda)$ is unitary when $\lambda \in \mathbb{R}\setminus\{0\}$.
Recall $S_{abs}(\lambda): g \mapsto f$ in context of (*).
So we want to have under (*),

$$\|g\|_{L^2} = \|f\|_{L^2}.$$

Let v be the solution to (*). Then for each $R > 0$,
$(P_V - \lambda^2)v = 0$, $(P_V - \lambda^2)\bar{v} = 0$

$$0 = \int_{B(0,R)} ((P_V - \lambda^2)v)\bar{v} - ((P_V - \lambda^2)\bar{v})v \, dx$$

$$= \int_{B(0,R)} v \cdot \Delta\bar{v} - \bar{v} \cdot \Delta v \, dx$$

$$= \oint_{\partial B(0,R)} v \cdot \partial_r \bar{v} - \bar{v} \cdot \partial_r v \, dS$$

$$= 2i \oint_{S^{n-1}} \operatorname{Im}\left(\left(e^{i\lambda R} f(\theta) + e^{-i\lambda R} g(\theta)\right) \cdot \left(-i\lambda e^{-i\lambda R}\overline{f(\theta)} + i\lambda e^{i\lambda R}\overline{g(\theta)}\right)\right) dS(\theta) + O(R^{-1}).$$

Letting $R \to \infty$, we get

$$0 = \operatorname{Re} \oint_{S^{n-1}} \left(e^{i\lambda R} f(\theta) + e^{-i\lambda R} g(\theta)\right)\left(e^{-i\lambda R}\overline{f(\theta)} - e^{i\lambda R}\overline{g(\theta)}\right) dS(\theta)$$

or equivalently $\int_{S^{n-1}} |f|^2 dS = \int_{S^{n-1}} |g|^2 dS$. $\square$

A brief overview of some other results:

I) Scattering determinant.

We have $S(\lambda) = I + A(\lambda)$, $A(\lambda)$ meromorphic ($\lambda \in \mathbb{C}$) and smoothing. In particular it is trace class

Then one can define $\det(S(\lambda))$.

See §§ B.3 – B.5.

$\det S(\lambda) = 0 \iff S(\lambda)$ is not invertible

(assuming λ not a resonance!)

$S(\lambda)$ not invertible $\Rightarrow$ the equation $(P_V - \lambda^2)v = 0$ has a nontrivial solution which is incoming at λ, i.e. ~~outgoing~~ at λ $\Rightarrow$ $-\lambda$ is a resonance!

In general have the ~~formula~~ (Thm 3.42):

$$m_S(\lambda) = m_R(\lambda) - m_R(-\lambda)$$

where $m_S(\lambda)$ = multiplicity of λ as a pole of $\det S(\lambda)$,

$$m_S(\lambda) = -\frac{1}{2\pi i} \oint_\lambda \partial_\zeta \log \det S(\zeta)\, d\zeta$$

$$= -\frac{1}{2\pi i} \operatorname{tr} \oint_\lambda S(\zeta)^{-1} \partial_\zeta S(\zeta)\, d\zeta .$$

and $m_R(\lambda)$ = multiplicity of λ as a resonance.

Note also: ~~$S(\lambda)$ unitary on~~ $S(\lambda)$ unitary for $\lambda \in \mathbb{R}$

$$\Downarrow$$

$$|\det S(\lambda)| = 1 \quad \text{for } \lambda \in \mathbb{R}.$$

II) So now resonances are related to singularities of $\det S(\lambda)$ which is just 1 function of $\lambda \in \mathbb{C}$ (!)

That makes it possible to use complex analysis to "express" $\det S(\lambda)$ in terms of resonances

(Melrose trace formula)

A consequence of this is the following fact (Thm 3.62):

$V \in C_c^\infty(\mathbb{R}^n; \mathbb{R})$, n odd, $V \not\equiv 0 \Rightarrow$

$\Rightarrow$ P_V has infinitely many resonances

III) On the other hand, Christiansen gave an example of $V \in C_c^\infty(\mathbb{R}^n; \mathbb{C})$, $n \geq 3$ odd, so that P_V has <u>no resonances</u>. (Thm 3.26)

IV) How many resonances are there? <u>Melrose, Zworski</u>:

<u>Thm 3.24</u>: $\#\{\lambda \text{ resonance}, |\lambda| \leq r\} = O(r^n)$ as $r \to \infty$.

V) Lower bounds on the number of resonances? in balls?

Not known except in dimension 1

<u>dim 1</u>: Thm 2.14 says that

$\#(\lambda \text{ res.}, |\lambda| \leq r) = \dfrac{2|\text{ch supp } V|}{\pi} r(1+o(1))$

with multiplicities

as $r \to \infty$, $|\text{ch supp } V|$ = diameter of supp V

VI) Lots of work on lower bounds for <u>generic</u> or <u>random</u> potentials (Sjöstrand, Christiansen...)