§6. Convolutions I

§6.1. Convolution with a smooth function

If $u \in L^1_{loc}(\mathbb{R}^n)$, $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have the convolution

$$u * \varphi(x) = \int_{\mathbb{R}^n} u(y)\varphi(x-y)\,dy$$

We now extend this to the case when u is a distribution!

<u>Defn</u> Let $u \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in C_c^\infty(\mathbb{R}^n)$. For each $x \in \mathbb{R}^n$, define

$$u * \varphi(x) = (u, \varphi(x - \cdot))$$

where $\varphi(x-\cdot) \in C_c^\infty(\mathbb{R}^n)$ is defined by

$$\varphi(x-\cdot)(y) = \varphi(x-y).$$

<u>Remark</u> Alternatively we could take $u \in \mathcal{E}'(\mathbb{R}^n)$, $\varphi \in C^\infty(\mathbb{R}^n)$.

<u>Example</u> $u = \delta_0 \in \mathcal{E}'(\mathbb{R}^n)$, $\varphi \in C^\infty(\mathbb{R}^n)$

$$\delta_0 * \varphi(x) = (\delta_0, \varphi(x-\cdot)) = \varphi(x)$$

I.e. $\delta_0 * \varphi = \varphi$.

Properties of convolution:

- $u * \varphi \in C^0(\mathbb{R}^n)$:

indeed, if $|x'| < \varepsilon < 1$ and $|x| \leq R$

$$|u*\varphi\,(x+x') - u*\varphi(x)| =$$
$$|(u, \varphi(x+x'-\cdot) - \varphi(x-\cdot))|$$
$$\leq C\,\|\varphi(x+x'-\cdot) - \varphi(x-\cdot)\|_{C^N}$$

for some C, N (where $\operatorname{supp} \varphi(x+x'-\cdot) - \varphi(x-\cdot) \subset B(0,R+1) - \operatorname{supp}\varphi \subset \mathbb{R}^n$ compact)

$$\leq C \max_{|\alpha| \leq N} \sup_{y \in \mathbb{R}^n} |\partial_y^\alpha(\varphi(x+x'-y) - \varphi(x-y))|$$

$\longrightarrow 0$ as $\varepsilon \to 0$

(in other words, $\varphi(x+x'-y) \to \varphi(x-y)$ in C_y^∞ as $x' \to 0$)

- $u * \varphi \in C^1(\mathbb{R}^n)$, and $\partial_{x_j}(u*\varphi) = u * \partial_{x_j}\varphi$:

if $|t| \leq 1$ and $|x| \leq R$ as above then

$$|u*\varphi(x+te_j) - u*\varphi(x) - t(u*\partial_{x_j}\varphi)(x)|$$
$$= |(u, \varphi(x+te_j-\cdot) - \varphi(x-\cdot) - t\partial_{x_j}\varphi(x-\cdot))| \leq$$
$$\leq C\,\|\varphi(x+te_j-\cdot) - \varphi(x-\cdot) - t\,\partial_{x_j}\varphi(x-\cdot)\|_{C^N}$$

$$\leq C \max_{|\alpha|\leq N} \sup_{y\in\mathbb{R}^n} \left|\partial_y^\alpha \left(\varphi(x+te_j-y)-\varphi(x-y)-t\partial_{x_j}\varphi(x-y)\right)\right|$$

$\leq C|t|^2$ as $t \to 0$.

- Iterating, we see:

$u \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi \in C_c^\infty(\mathbb{R}^n)$ $\Rightarrow$

$\Rightarrow u*\varphi \in C^\infty(\mathbb{R}^n)$ and $\forall \alpha$,

$$\partial^\alpha(u*\varphi) = u*\partial^\alpha\varphi.$$

Note also $u*\partial^\alpha\varphi = \partial^\alpha u*\varphi$ by the definition of derivative in distributions:

$$u*\partial_{x_j}\varphi = (u, (\partial_{x_j}\varphi)(x-\cdot))$$
$$- (u(y), \partial_{y_j}(\varphi(x-y)))$$
$$= (\partial_{y_j}u(y), \varphi(x-y))$$
$$= \partial_{y_j}u*\varphi$$

Remark on convergence in $\mathcal{D}'$

(not really a good place here but whatever)

Assume $u_k \in \mathcal{D}'(U)$ and $\forall \varphi \in C_c^\infty(U)$, the sequence (u_k, φ) is bounded.

Then $\forall K \subset U$ compact $\exists C, N$ such that $\forall \varphi \in C_c^\infty(U)$, $\operatorname{supp} \varphi \subset K$ $\forall k$ we have

$$|(u_k, \varphi)| \le C \|\varphi\|_{C^N}.$$

__Proof__: We can reduce to $u_k \in \mathcal{E}'(U)$ (multiply by a cutoff) and $\forall \varphi \in C^\infty(U)$, $k \mapsto (u_k, \varphi)$ is bdd.

Define the sets $\forall L \in \mathbb{N}$

$$A_L := \{\varphi \in C^\infty(U) : \forall k, |(u_k, \varphi)| \le L\}.$$

Then $C^\infty(\bar{U}) = \bigcup_{L \in \mathbb{N}} A_L$.

Since $C^\infty(\bar{U})$ is a complete metric space, by the Baire category theorem

$\exists L$: interior (closure $(A_L)) \neq \emptyset$.

But each A_L is closed in $C^\infty(\bar{U})$:

if $\varphi_j \to \varphi$ in $C^\infty(\bar{U})$

then $(u_k, \varphi_j) \xrightarrow[j\to\infty]{} (u_k, \varphi)$ $\forall k$

So we can fix L such that

interior $(A_L) \neq \emptyset$,

i.e. A_L contains a ball $\overline{B(\psi, \varepsilon)}$

for some $\varepsilon > 0$, $\psi \in C^\infty(\bar{U})$.

Since $d(\varphi, 0) = d(\psi, \varphi + \psi)$

we get:

$\forall \varphi$, if $d(\varphi, 0) \leq \varepsilon$ then

$\varphi + \psi \in A_L$.

Also $\psi \in A_L$, so $\varphi \in A_{2L}$.

So: $\forall \varphi \in C^\infty(\bar{U})$, if $d(\varphi, 0) < \varepsilon$ then $\forall k$, $|(u_k, \varphi)| \leq 2L$.

Recalling $d(\varphi, 0) = \sum_N 2^{-N} \frac{\|\varphi\|_{C^N(K_N)}}{1 + \|\varphi\|_{C^N(K_N)}}$

we see $\exists N, \delta > 0$ such that

$$\|\varphi\|_{C^N(K_N)} \leq \delta \Rightarrow d(\varphi, 0) < \varepsilon \Rightarrow \forall k, \; |(u_k, \varphi)| \leq 2L.$$

Rescaling we see that $\exists C \left(= \frac{2L}{\delta}\right)$ such that $\forall \varphi \in C^\infty(\bar{U})$, $\forall k$

$$|(u_k, \varphi)| \leq C \cdot \|\varphi\|_{C^N(K_N)}$$

which gives the needed uniform bound. $\square$

<u>Corollary</u>: if $u_k \to u$ in $\mathcal{D}'(U)$, $\varphi_k \to \varphi$ in $C_c^\infty(U)$

Then $(u_k, \varphi_k) \to (u, \varphi)$

<u>Proof</u> $|(u_k, \varphi_k) - (u, \varphi)| \leq |(u_k, \varphi_k - \varphi)| + |(u_k - u, \varphi)|$.

First term goes to 0 by the estimate above, 2nd term goes to 0 since $u_k \to u$ in $\mathcal{D}'$.

Back to convolutions: we have sequential continuity:

- if $u_k \to u$ in $\mathcal{D}'(\mathbb{R}^n)$ and $\varphi_k \to \varphi$ in $C_c^\infty(\mathbb{R}^n)$ then $u_k * \varphi_k \to u * \varphi$ in $C^\infty(\mathbb{R}^n)$

<u>Proof</u> Enough to show

$$u_k * \varphi_k(x) \longrightarrow u * \varphi(x)$$

locally uniformly in x.

That is, need to show that $\forall$ sequence $x_k \to x$ we have $u_k * \varphi_k(x_k) \to u * \varphi(x)$.

(lack of local uniformity gives a counterexample sequence...)

But $u_k * \varphi_k(x_k) = (u_k, \varphi_k(x_k - \cdot))$

and $u_k \to u$ in $\mathcal{D}'(\mathbb{R}^n)$

$$\varphi_k(x_k - \cdot) \to \varphi(x - \cdot) \text{ in } C_c^\infty(\mathbb{R}^n)$$

So $u_k * \varphi_k(x_k) \longrightarrow (u, \varphi(x - \cdot)) = u * \varphi(x)$ as needed.

§6.2. Approximation by smooth fns.

Here we show ($U \subset \mathbb{R}^n$ open)

Thm $C_c^\infty(U)$ is dense in $\mathcal{D}'(U)$

i.e. $\forall u \in \mathcal{D}'(U)$ $\exists u_n \in C_c^\infty(U)$

s.t. $u_n \to u$ in $\mathcal{D}'(U)$.

The main step is the following

Proposition Let $u \in \mathcal{D}'(\mathbb{R}^n)$.

Fix $\chi \in C_c^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \chi = 1$

For $\varepsilon > 0$ define $\chi_\varepsilon(x) := \varepsilon^{-n} \chi\left(\frac{x}{\varepsilon}\right)$

and put $u_\varepsilon := u * \chi_\varepsilon \in C^\infty(\mathbb{R}^n)$.

Then $u_\varepsilon \to u$ in $\mathcal{D}'(\mathbb{R}^n)$.

Proof of Proposition Take $\varphi \in C_c^\infty(\mathbb{R}^n)$.

We need $(u_\varepsilon, \varphi) \to (u, \varphi)$.

If $u \in L^1_{loc}(\mathbb{R}^n)$ then

$$(u_\varepsilon, \varphi) = \int_{\mathbb{R}^n} u_\varepsilon(x)\varphi(x)\,dx$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(y)\chi_\varepsilon(x-y)\,\varphi(x)\,dy\,dx$$

$$= \int_{\mathbb{R}^n} u(y)\varphi_\varepsilon(y)\,dy = (u, \varphi_\varepsilon) \text{ where}$$

$$\varphi_\varepsilon(y) := \int_{\mathbb{R}^n} \chi_\varepsilon(x-y)\varphi(x)\,dx \in C_c^\infty(\mathbb{R}^n).$$

We claim that for any $u \in \mathcal{D}'(\mathbb{R}^n)$

$$\boxed{(*)\quad (u_\varepsilon, \varphi) = (u, \varphi_\varepsilon).}$$

But first $(*) \Rightarrow$ Proposition:

we have $\varphi_\varepsilon \to \varphi$ in C_c^∞:

$$\varphi_\varepsilon(y) = \int_{\mathbb{R}^n} \chi_\varepsilon(x)\varphi(x+y)\,dx$$

$$\varphi(y) = \int_{\mathbb{R}^n} \chi_\varepsilon(x)\varphi(y)\,dx$$

$$\partial^\alpha(\varphi_\varepsilon - \varphi)(y) = \int_{\mathbb{R}^n} \chi_\varepsilon(x)\left(\partial^\alpha\varphi(x+y) - \partial^\alpha\varphi(y)\right)dy$$

goes to 0 uniformly in y

Since u is a distribution,

$$(u_\varepsilon, \varphi) = (u, \varphi_\varepsilon) \xrightarrow[\varepsilon \to 0]{} (u, \varphi)$$

which gives $u_\varepsilon \to u$ in $\mathcal{D}'$.

Now, <u>proof of (★)</u>:

need to show that

$$\int_{\mathbb{R}^n} u_\varepsilon(x)\varphi(x)\,dx = (u, \varphi_\varepsilon) \quad \text{where}$$

$$\varphi_\varepsilon(y) = \int_{\mathbb{R}^n} \chi_\varepsilon(x-y)\varphi(x)\,dx \in C_c^\infty(\mathbb{R}^n),$$

$$u_\varepsilon(x) = (u, \varphi(x - \cdot)) \in C^\infty(\mathbb{R}^n)$$

Formally we have

$$\varphi_\varepsilon = \int_{\mathbb{R}^n} \chi_\varepsilon(x - \cdot)\varphi(x)\,dx$$

and we need

$$(u, \varphi_\varepsilon) = \int_{\mathbb{R}^n} (u, \chi_\varepsilon(x - \cdot))\varphi(x)\,dx.$$

(roughly speaking, u can be put under the $\int$ sign)

To prove this, we use

Riemann sums:

for $\psi \in C_c^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \psi \, dx = \lim_{\delta \to 0+} \delta^n \sum_{x \in \delta \cdot \mathbb{Z}^n} \psi(x).$$

We have now

$$\varphi_\varepsilon(y) = \lim_{\delta \to 0+} \delta^n \sum_{x \in \delta \cdot \mathbb{Z}^n} \chi_\varepsilon(x-y)\,\varphi(x)$$

where the sum is finite for each $\delta > 0$ and the $\lim_{\delta \to 0+}$ converges in $C_c^\infty(\mathbb{R}^n)$ (in the y variable)

So

$$(u, \varphi_\varepsilon) = \lim_{\delta \to 0+} \delta^n \sum_{x \in \delta \cdot \mathbb{Z}^n} (u, \chi_\varepsilon(x - \cdot))\,\varphi(x)$$

$$= \lim_{\delta \to 0+} \delta^n \sum_{x \in \delta \cdot \mathbb{Z}^n} u_\varepsilon(x)\,\varphi(x) = \int_{\mathbb{R}^n} u_\varepsilon \varphi \, dx$$

proving $(\star)$
and finishing the proof of Proposition.

$\square$

Now let's prove the <u>Thm</u>

($C_c^\infty(U)$ dense in $\mathcal{D}'(U)$).

Take $u \in \mathcal{D}'(U)$
and χ_ε as in Proposition.

Take a family of compact sets
$K_1 \subset K_2 \subset \cdots \subset U$, $U = \bigcup_{\ell \in \mathbb{N}} \mathring{K}_\ell$

and fix cutoffs
$\psi_\ell \in C_c^\infty(U)$, $\operatorname{supp}(1-\psi_\ell) \cap K_\ell = \emptyset$.

(Note: $\forall$ compact $K \subset U$, $\operatorname{supp}(1-\psi_\ell) \cap K = \emptyset$ for ℓ large enough)

Now define
$u_\ell := (\psi_\ell u) * \chi_{\varepsilon_\ell}$ where $\varepsilon_\ell \downarrow 0$ is chosen below

Here $\psi_\ell u \in \mathcal{E}'(U)$
and we extend it by 0
to $\psi_\ell u \in \mathcal{E}'(\mathbb{R}^n)$ (see Pset 3)

If $\operatorname{supp} \chi \subset B(0,1)$ then
$\operatorname{supp} (\psi_\ell u) * \chi_{\varepsilon_\ell} \subset \operatorname{supp} \psi_\ell + B(0, \varepsilon_\ell) \subset U$ if ε_ℓ small enough

So then $u_\ell \in C_c^\infty(U)$.

Now $u_\ell \to u$ in $\mathcal{D}'(U)$:

fix $\varphi \in C_c^\infty(U)$. Then extend it to $\varphi \in C_c^\infty(\mathbb{R}^n)$.

We have $(u_\ell, \varphi) = ((\psi_\ell u) * \chi_{\varepsilon_\ell}, \varphi)$.

But actually

$$\boxed{(u_\ell, \varphi) = ((\psi_{\ell_0} u) * \chi_{\varepsilon_\ell}, \varphi), \quad \ell \geq \ell_0}$$

where ℓ_0 depends only on φ!

we need: $(((\psi_\ell - \psi_{\ell_0})u) * \chi_{\varepsilon_\ell}, \varphi) = 0$

enough: $(\operatorname{supp}(\psi_\ell - \psi_{\ell_0}) + B(0, \varepsilon_\ell)) \cap \operatorname{supp} \varphi = \emptyset$

i.e. $\operatorname{supp}(\psi_\ell - \psi_{\ell_0}) \cap (\operatorname{supp} \varphi + B(0, \varepsilon_\ell)) = \emptyset$.

Now $\operatorname{supp}(\psi_\ell - \psi_{\ell_0}) \cap K_{\ell_0} = \emptyset$

So enough $\operatorname{supp} \varphi + B(0, \varepsilon_{\ell_0}) \subset K_{\ell_0}$

which is true for ℓ_0 large enough depending on $\operatorname{supp} \varphi$

Now

$$(u_\ell, \varphi) = ((\psi_{\ell_0} u) * \chi_{\varepsilon_\ell}, \varphi) \longrightarrow$$

$$\to (\psi_{\ell_0} u, \varphi) \quad \text{by Proposition}$$

$$= (u, \varphi) \quad \text{since } \psi_{\ell_0} = 1 \text{ on } \operatorname{supp} \varphi.$$

$\square$