§17. A bit of spectral theory

§17.1. A spectral theorem

Here we prove

Thm Assume M is a compact manifold, $P \in \mathrm{Diff}^m(M)$ is elliptic and formally self-adjoint (i.e. $P^* = P$, or equivalently $\langle P\varphi, \psi \rangle_{L^2} = \langle \varphi, P\psi \rangle_{L^2}$ $\forall \varphi, \psi \in C^\infty(M)$) and $m > 0$.

Then $\exists$ a sequence $u_k \in C^\infty(M)$ & a sequence $\lambda_k \in \mathbb{R}$, s.t.

- $P_k u_k = \lambda_k u_k$
- $|\lambda_k| \to \infty$
- $\{u_k\}$ forms an orthonormal (Hilbert) basis of $L^2(M)$, in particular we have "Fourier series": $f \in L^2(M) \Leftrightarrow$ $\Leftrightarrow f = \sum_k c_k u_k$ in $L^2(M)$ where $\sum_k |c_k|^2 < \infty$.

Remarks

1. A fundamental example is the Laplacian $-\Delta_g$ on a compact Riemannian manifold (M,g)

Note that $\lambda_k \geq 0$ in this case:

$$\lambda_k = \langle \lambda_k u_k, u_k \rangle_{L^2} = \langle -\Delta_g u_k, u_k \rangle_{L^2}$$
$$= \int_M |\nabla_g u_k|^2 \, dVol_g \geq 0$$

2. Can use Thm to get solutions to evolution equations. E.g. the wave equation:

$$\begin{cases} (\partial_t^2 - \Delta_g) u(t,x) = 0, & t \geq 0, \ x \in M \\ u|_{t=0} = f_0(x) \\ u_t|_{t=0} = f_1(x). \end{cases}$$

If $f_0(x) = \sum_k f_{0,k} \cdot u_k(x)$, $f_1(x) = \sum_k f_{1,k} u_k(x)$

then $$u(t,x) = \sum_k \left(f_{0,k} \cos(\sqrt{\lambda_k} t) + f_{1,k} \frac{\sin(\sqrt{\lambda_k} t)}{\sqrt{\lambda_k}} \right) \cdot u_k(x).$$

Proof

1. For each $\lambda \in \mathbb{R}$, the operator

$$P-\lambda = P-\lambda I : H^m(M) \to L^2(M)$$

is Fredholm.

Indeed, since $m>0$, $P-\lambda \in \mathrm{Diff}^m(M)$ has same principal symbol as P and thus is elliptic.

Moreover, the index of $P-\lambda$ is equal to 0 since $(P-\lambda)^* = P-\lambda$ and $\mathrm{ind}\,(P-\lambda)^* = -\,\mathrm{ind}\,(P-\lambda)$

So, either $P-\lambda$ is invertible $H^m \to L^2$ or the eigenspace

$$E_\lambda := \{ u \in C^\infty(M) \mid Pu = \lambda u \}$$

is nontrivial, but $\dim E_\lambda < \infty$.

Define the spectrum

$$\mathrm{Spec}(P) = \{ \lambda \in \mathbb{R} \mid E_\lambda \neq \{0\} \}.$$

2. We next show that the set $\mathrm{Spec}(P)$ is <u>discrete</u>: if $\lambda \in \mathrm{Spec}(P)$ then $\exists \varepsilon > 0$:

$$(\lambda - \varepsilon, \lambda + \varepsilon) \cap \mathrm{Spec}\, P = \{\lambda\}.$$

Indeed, define the orthogonal complements

$$H^m_\perp := \{u \in H^m(M) \mid \forall v \in E_\lambda,\ \langle u, v\rangle_{L^2} = 0\}$$

$$L^2_\perp := \{u \in L^2(M) \mid \forall v \in E_\lambda,\ \langle u, v\rangle_{L^2} = 0\}.$$

Then $P - \lambda : H^m_\perp \to L^2_\perp$ is invertible:

- Since $H^m = H^m_\perp \oplus E_\lambda$, we have

$$(P-\lambda) H^m_\perp = (P-\lambda) H^m = L^2_\perp$$

Since $\mathrm{Range}(P-\lambda)$ = orthogonal complement of $\ker(P-\lambda)^*$ and $(P-\lambda)^* = (P-\lambda)$

- $P - \lambda : H^m_\perp \to L^2_\perp$ is injective, as $H^m_\perp \cap E_\lambda = \{0\}$
- By Banach's Thm, $(P-\lambda)^{-1} : L^2_\perp \to H^m_\perp$ is a bounded operator

Now $\exists\, \varepsilon > 0\ \forall \lambda' \in (\lambda - \varepsilon, \lambda + \varepsilon)$,
the operator $\underline{P} - \lambda' : H^m_\perp \to L^2_\perp$
is invertible.
If $\lambda' \neq \lambda$ then $\underline{P} - \lambda' : E_\lambda \to E_\lambda$
is also invertible: it's equal to $(\lambda - \lambda')\underline{I}$.
Since $H^m = H^m_\perp \oplus E_\lambda$, $L^2 = L^2_\perp \oplus E_\lambda$
we see that $\underline{P} - \lambda' : H^m \to L^2$
is invertible, so $\lambda' \notin \mathrm{Spec}(\underline{P})$.

3. If $\lambda \neq \lambda'$ are in $\mathrm{Spec}(\underline{P})$,
then $E_\lambda \perp E_{\lambda'}$ in L^2.
Indeed, $\forall u_\lambda \in E_\lambda$, $u_{\lambda'} \in E_{\lambda'}$

$$\langle P u_\lambda, u_{\lambda'} \rangle_{L^2} = \langle u_\lambda, P u_{\lambda'} \rangle_{L^2}$$

$$\lambda \langle u_\lambda, u_{\lambda'} \rangle_{L^2} = \lambda' \langle u_\lambda, u_{\lambda'} \rangle_{L^2}.$$

So $\exists$ an orthonormal system
consisting of orthonormal bases of all E_λ.

4. It remains to show that the above orthonormal system is complete. That is, we need to show that the orthogonal complement

$$V := \Big(\bigoplus_{\lambda \in \mathrm{Spec}(P)} E_\lambda \Big)^\perp = \{ u \in L^2(M) : u \perp E_\lambda \ \forall \lambda \}$$

is equal to $\{0\}$.

WLOG $0 \notin \mathrm{Spec}(P)$

(can replace P by $P - \lambda_0$ for some $\lambda_0 \notin \mathrm{Spec}(P)$)

Then P is invertible $H^m \to L^2$, with the inverse $P^{-1} : L^2 \to H^m$.

We can think of P^{-1} as an operator $L^2 \to L^2$, then

$\boxed{1}$ $P^{-1} : L^2 \circlearrowleft$ is compact

$\boxed{2}$ P^{-1} is self-adjoint because P is:

$\forall u, v \in L^2$, $\langle P^{-1} u, v \rangle_{L^2} = \langle P^{-1} u, P P^{-1} v \rangle$

$= \langle P P^{-1} u, P^{-1} v \rangle = \langle u, P^{-1} v \rangle$

(here we use that $\langle Pf, g\rangle = \langle f, Pg\rangle$ $\forall f, g \in H^m$)

$\boxed{3}$ $\underline{P}^{-1} : V \to V$. Indeed, if $u \in V$ and $v \in E_\lambda$ for some λ then $\langle P^{-1}u, v\rangle_{L^2} = \langle u, P^{-1}v\rangle_{L^2}$

$= \lambda^{-1}\langle u, v\rangle_{L^2} = 0$, so $P^{-1}u \in V$.

$\boxed{4}$ $\underline{P}^{-1}|_V$ has no (real) eigenvalues: indeed, if $\mu \in \mathbb{R}$ and $u \in V$ satisfy $u \neq 0$, $\underline{P}^{-1}u = \mu \cdot u$, then $\mu \neq 0$ (as $\underline{P}\,\underline{P}^{-1}u = u$), $u \in H^m$, and $Pu = \mu^{-1}u$, so $u \in E_{\mu^{-1}}$, which is impossible as $V \perp E_\lambda$ $\forall \lambda \in \mathrm{Spec}(P)$.

Now if $V \neq \{0\}$ then $\boxed{1}$ – $\boxed{4}$ cannot all be true.

This follows by applying to $P^{-1}|_V$ the Thm on the next page. $\square$

Thm [Hilbert–Schmidt]

Assume $A : V \to V$ is a compact self-adjoint operator on a Hilbert space V and A is not identically 0.
Then A has a nonzero eigenvalue.

Proof 1 $\|A\| = \sup_{\substack{u \in V \\ \|u\|=1}} |\langle Au, u\rangle|$.

Indeed, $\geq$ is immediate.
For $\leq$, use the identity (using that $A^* = A$)

$$\langle A(u+v), u+v\rangle - \langle A(u-v), u-v\rangle$$
$$= 4\operatorname{Re}\langle Au, v\rangle$$ to get,

with $r := \sup_{\substack{u \in V \\ \|u\|=1}} |\langle Au, u\rangle|$,

$$4\operatorname{Re}\langle Au, v\rangle \leq r\left(\|u+v\|^2 + \|u-v\|^2\right)$$
$$= 2r\left(\|u\|^2 + \|v\|^2\right)$$

Put $v := tAu$ for some $t > 0$,
then $4t\|Au\|^2 \leq 2r(\|u\|^2 + t^2\|Au\|^2)$

Putting $t := \frac{\|u\|}{\|Au\|}$, get

$$4\|u\| \cdot \|Au\| \leq 4r\|u\|^2 \Rightarrow \|Au\| \leq r\|u\|$$

2 Since $A \neq 0$, we know that

$$r = \|A\| = \sup_{\|u\|=1} |\langle Au, u\rangle| > 0.$$

Take a sequence u_k : $\|u_k\| = 1$,

$$\langle Au_k, u_k\rangle \longrightarrow r \quad \text{or} \quad -r$$

WLOG the limit is r (can do $A \to -A$)

Since A is compact, passing to a subsequence can make

$$Au_k \to v \text{ for some } v \in V.$$

We now claim that $v \neq 0$ and

$$Av = rv,$$ i.e. r is an eigenvalue of A:

$$\|Au_k - ru_k\|^2 =$$
$$= \|Au_k\|^2 - 2r\langle Au_k, u_k\rangle + r^2\|u_k\|^2$$
$$\le r^2 - 2r\langle Au_k, u_k\rangle + r^2$$
$$= 2r^2 - 2r\langle Au_k, u_k\rangle \xrightarrow[k\to\infty]{} 0$$

So $Au_k - ru_k \to 0$.

But $Au_k \to v$, so $ru_k \to v$.

Thus $u_k \to r^{-1}v$. This implies that $Au_k \to Ar^{-1}v = v$, so $Av = rv$ as needed. $\square$

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§17.2. Various results on Δ_g

Assume (M, g) is a compact Riemannian manifold.

Look at the spectrum of $-\Delta_g$:

$$-\Delta_g u_k = \lambda_k u_k$$

$$0 = \lambda_1 < \lambda_2 \leq \dots, \quad \lambda_k \to \infty$$

$u_1, u_2, \dots \in C^\infty(M)$
orthonormal basis of $L^2(M)$

One can ask a lot of questions on the behavior of λ_k and u_k as $k \to \infty$.
Here we discuss some results.
No proofs are given in this section.

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1. Weyl Law: if $\dim M = n$

$$N(R) = \#\{k : \lambda_k \leq R^2\}$$

Then as $R \to \infty$ $\quad (\omega_n := \mathrm{Vol}(B_{\mathbb{R}^n}(0,1)))$

$$N(R) = (2\pi)^{-n} \omega_n \mathrm{Vol}_g(M) R^n + O(R^{n-1})$$

Goes back to Weyl 1911
(domains in $\mathbb{R}^n$)
In the setting stated above:
Levitan 1952, 1955
Avakumović 1956

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② Better remainder in Weyl Law:

Can we improve $O(R^{n-1})$?

In general, NO:
if $M = S^2$ is the round 2-sphere
then it has eigenvalues $\ell(\ell+1)$,
$\ell = 0, 1, \ldots,$ with multiplicities $2\ell+1$.
If $R_\ell = \sqrt{\ell(\ell+1)}$ then

$$N(R_\ell + \varepsilon) - N(R_\ell - \varepsilon) = 2\ell+1 \sim R_\ell$$

So cannot get $N(R) \sim R^2 + o(R)$.

But typically, YES:

if the set of closed geodesics on (M, g) has measure 0 (as a subset of the tangent bundle TM)

then $N(R) = (2\pi)^{-n} \omega_n \mathrm{Vol}_g(M) R^n + o(R^{n-1})$

This was proved in Duistermaat – Guillemin 1975

Open problem: if M is negatively curved, can we get $O(R^{n-1-\varepsilon})$ for some $\varepsilon > 0$?

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③ Better remainder when M has a boundary (and we study Dirichlet eigenvalues: $u_k|_{\partial M} = 0$):

Weyl's conjecture:

$$N(R) = (2\pi)^{-n} \omega_n \mathrm{Vol}_g(M) R^n - \frac{(2\pi)^{1-n}}{4} \omega_{n-1} \mathrm{Vol}_g(\partial M) R^{n-1} + o(R^{n-1})$$

Assuming the set of closed billiard geodesics has measure 0,

this was proved by

Melrose 1980 if ∂M is strictly concave

Ivrii 1980 for any C^∞ boundary

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4 Nodal sets: take u_k real valued.

What is the asymptotic of $A(\lambda_k) = \text{Area}(\{x \in M : u_k(x) = 0\})$?

Yau's conjecture: $\exists c, C \ \forall k$

$$c\sqrt{\lambda_k} \leq A(\lambda_k) \leq C\sqrt{\lambda_k}.$$

Still open in general but

Donnelly - Fefferman 1988: true for real analytic (M, g)

Colding - Minicozzi 2011:

$$A(\lambda_k) \geq c\, \lambda_k^{\frac{3-n}{4}}, \quad n = \dim M$$

Logunov 2018:

$$c\sqrt{\lambda_k} \le A(\lambda_k) \le C\lambda_k^{C_n}$$

C_n constant depending only on n

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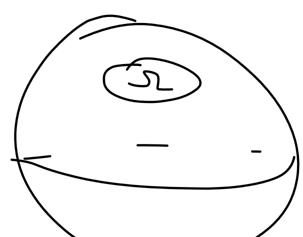
5 Lower bounds on mass:

Thm Assume (M,g) is either $\mathbb{T}^n$ or a negatively curved surface.

Then $\forall$ nonempty open $\Omega \subset M$ $\exists\, c_\Omega > 0$ $\forall k$

$$\|u_k\|_{L^2(\Omega)} \geq c_\Omega.$$

Remark: this fails for the sphere $\mathbb{S}^2$: for some Ω e.g.



we have

$$\|u_k\|_{L^2(\Omega)} \sim e^{-C\sqrt{\lambda_k}}.$$

(Unique continuation gives the lower bound $\forall M$)

For $\mathbb{T}^n$: Jaffard 1990, Haraux 1989, Komornik 1992

For negatively curved surfaces:

Dyatlov-Jin-Nonnenmacher 2021

using Bourgain-Dyatlov 2018

Open problem: does this hold for (M,g) negatively curved of dim ≥ 3?