18.155 LEC 15

§15. Variable coefficient elliptic PDE

§15.1. Motivation: Elliptic Regularity III

Let M be a manifold,

$P \in \mathrm{Diff}^m(M)$ a differential operator,

$p := \sigma_m(P) \in C^\infty(T^*M)$ its principal symbol.

Defn. We say P is *elliptic* if the equation $p(x,\xi) = 0$ has no solutions $(x,\xi) \in T^*M$ with $\xi \neq 0$.

Example: the Laplace-Beltrami operator $\Delta_g \in \mathrm{Diff}^2$ of some Riemannian metric g on M

Here $\sigma_2(\Delta_g)(x,\xi) = -|\xi|^2_{g(x)}$

Thm. [Elliptic Regularity III]
Assume P is elliptic. Then $\forall u \in \mathcal{D}'(M)$ we have

$$\operatorname{sing\,supp} u \subset \operatorname{sing\,supp}(Pu)$$

In particular, $Pu \in C^\infty(M) \Rightarrow u \in C^\infty(M)$.

§15.2. Pseudodifferential Operators

(a brief exposition: might be done better in 18.157]

The key ingredient in proving Elliptic Regularity III will be the construction of an elliptic parametrix as a pseudodifferential operator.

We introduce these here.

Let $U \subset \mathbb{R}^n$ be open.

Defn. For $\ell \in \mathbb{R}$, define the space of Kohn-Nirenberg symbols of order ℓ, denoted $S^\ell(U \times \mathbb{R}^n)$, which contains all functions

$a(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$ such that

$\forall$ compact set $K \subset U$ and

$\forall$ multiindices α, β $\exists$ constant $C_{\alpha\beta K}$

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta K} \langle \xi \rangle^{\ell - |\beta|}$$

$\forall x \in K$, $\xi \in \mathbb{R}^n$.

$\left(\langle \xi \rangle := \sqrt{1+|\xi|^2}\right)$

Remarks:

1. Roughly speaking, $a \in S^{\ell}(U \times \mathbb{R}^n)$ if $a(x,\xi) = O(\langle \xi \rangle^{\ell})$ locally in x and differentiating in x gives same bound, differentiating in ξ improves by $\langle \xi \rangle^{-1}$

2. We have $S^{\ell} \subset S^{\ell'}$ when $\ell \le \ell'$ and the intersection $\bigcap_{\ell \in \mathbb{R}} S^{\ell}(U \times \mathbb{R}^n)$ is equal to the space of <u>rapidly decaying symbols</u>

$$S^{-\infty}(U \times \mathbb{R}^n) = \{ a \in C^{\infty}(U \times \mathbb{R}^n) : \forall \alpha, \beta, N \quad \partial_x^{\alpha} \partial_{\xi}^{\beta} a(x,\xi) = O(\langle \xi \rangle^{-N}) \text{ locally uniformly in } x \}$$

3. If $a = \sum_{|\alpha| \le m} a_{\alpha}(x) \xi^{\alpha}$, $a_{\alpha} \in C^{\infty}(U)$ then $a \in S^m(U \times \mathbb{R}^n)$ (symbol of a differential operator)

Defn. Assume that $\ell \in \mathbb{R}$, $a \in S^{\ell}(U \times \mathbb{R}^n)$. Define the operator $Op(a): C_c^\infty(U) \to C^\infty(U)$ by

$$\boxed{Op(a)\varphi(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\hat{\varphi}(\xi)\,d\xi}$$

$\forall \varphi \in C_c^\infty(U)$

The class of all such $Op(a)$ is called pseudodifferential operators of order ℓ.

Properties 1 Why $Op(a): C_c^\infty(U) \to C^\infty(U)$?

If $\varphi \in C_c^\infty(U)$ then $\hat{\varphi} \in S(\mathbb{R}^n)$.

Since $a(x,\xi) = O(\langle\xi\rangle^{\ell})$, the integral converges

And differentiating in x just gives extra powers of ξ...

2 The transpose $Op(a)^t$ also acts $C_c^\infty \to C^\infty$ (see Pset 10), so $Op(a): \mathcal{E}'(U) \to \mathcal{D}'(U)$.

3 If $a(x,\xi)\equiv 1$:

$$Op(1) = \text{Identity Operator } C_c^\infty(U) \circlearrowleft$$

by the Fourier Inversion Formula

4 If $a(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$:

$$Op(a) = \sum_{|\alpha|\leq m} a_\alpha(x) D_x^\alpha, \quad D_x = -i\partial_x$$

(see Pset 10)

5 If $a(x,\xi) = b(\xi)$, $U = \mathbb{R}^n$:

$$\widehat{Op(a)\varphi}(\xi) = b(\xi)\hat{\varphi}(\xi),$$ i.e.

$Op(a)$ is a Fourier multiplier.

It's also a convolution operator:

$$Op(a)\varphi = E * \varphi \text{ where}$$

$$E \in \mathcal{S}'(\mathbb{R}^n), \quad \hat{E}(\xi) = b(\xi)$$

(those were featured in the proof of Elliptic Regularity II in §12.2)

6 If $a \in S^{-\infty}(U \times \mathbb{R}^n)$ then $\mathrm{Op}(a)$ is <u>smoothing</u>, i.e. extends to

$$\mathrm{Op}(a) : \mathcal{E}'(U) \to C^\infty(U)$$

(sequentially continuous).

Indeed, for each $\varphi \in C_c^\infty(U)$ we have by Fubini

$$\mathrm{Op}(a)\varphi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi) \hat{\varphi}(\xi) d\xi$$

$$= (2\pi)^{-n} \int_{U_y \times \mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} a(x,\xi) \varphi(y)\, dy\, d\xi.$$

$$= \int_U K(x,y) \varphi(y) dy \quad \text{where}$$

the Schwartz kernel K is

$$K(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,\xi) d\xi$$

and $K(x,y) \in C^\infty(U \times U)$ because $a(x,\xi)$ is C^∞ in x with all derivatives decaying rapidly in ξ. Now see Pset 5, Problem 1

7 In general the Schwartz kernel of $Op(a)$, $a \in S^{\ell}(U \times \mathbb{R}^n)$, is given by $k \in \mathcal{D}'(U \times U)$ defined as follows:

Let $\check{a}(x, z)$ be the inverse Fourier transform of $a(x, \xi)$ in $\xi \to z$ variable, i.e. for $a \in S^{-\infty}$ we have

$$\check{a}(x,z) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iz\cdot\xi} a(x,\xi)\, d\xi$$

and in general $\check{a}$ is C^∞ in $x \in U$ with values in S' in z.

More precisely, $\check{a} \in \mathcal{D}'(U \times \mathbb{R}^n)$ and $\forall\, \varphi \in C_c^\infty(U \times \mathbb{R}^n)$ we have

informally

$$(\check{a}, \varphi) = \text{"}(2\pi)^{-n} \int_{U \times \mathbb{R}^{2n}} e^{iz\cdot\xi} a(x,\xi)\, \varphi(x,z)\, dz\, dx\, d\xi\text{"}$$

$$= (2\pi)^{-n} \int_{U \times \mathbb{R}^n} a(x,\xi) \cdot \overbrace{\left(\int_{\mathbb{R}^n} e^{iz\cdot\xi} \varphi(x,z)\, dz \right)}^{\text{rapidly decaying in } \xi} dx\, d\xi$$

Then K is

pull back by $(x,y)\in U\times U \mapsto (x,x-y)\in U\times\mathbb{R}^n$

$$\boxed{K(x,y) = \check{a}(x, x-y)}$$

This is immediate when $a\in S^{-\infty}$.

In general: need to check that $\forall \varphi,\psi\in C_c^\infty(U)$ we have

$$(\mathrm{Op}(a)\varphi,\psi) = (K(x,y),\psi(x)\varphi(y)).$$

The right-hand side is

$$(\check{a}(x,x-y),\psi(x)\varphi(y)) = \quad \text{(change of variables } z=x-y\text{)}$$

$$= (\check{a}(x,z),\psi(x)\varphi(x-z))$$

$$= (2\pi)^{-n}\int_{U\times\mathbb{R}^n} a(x,\xi)\Big(\int_{\mathbb{R}^n} e^{iz\cdot\xi}\psi(x)\varphi(x-z)\,dz\Big)\,dx\,d\xi$$

$$= (2\pi)^{-n}\int_{U\times\mathbb{R}^n} a(x,\xi)\psi(x)\Big(\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\varphi(y)\,dy\Big)\,dx\,d\xi$$

$$= (2\pi)^{-n}\int_{U\times\mathbb{R}^n} a(x,\xi)\psi(x)e^{ix\cdot\xi}\hat{\varphi}(\xi)\,dx\,d\xi$$

$$= (\mathrm{Op}(a)\varphi,\psi) \text{ indeed.}$$

8 We claim that

$$\operatorname{Sing\,supp} K \subset \text{diagonal } \{(x,x) \mid x \in U\}$$

Indeed, it suffices to show that $\forall j$, K is smooth on the open set

$$\{(x,y) \in U \times U \mid x_j \neq y_j\}.$$

We have $K(x,y) = \check{a}(x, x-y)$, so $\forall N$

$$(x_j - y_j)^N K(x,y) = (z_j^N \check{a})(x, x-y)$$

And since $\check{a}(x,z) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iz\cdot\xi} a(x,\xi)\, d\xi$

formally,

we can check formally that

$$z_j^N \check{a}(x,z) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left((-i\partial_{\xi_j})^N e^{iz\cdot\xi} \right) a(x,\xi)\, d\xi$$

(int. by parts N times)

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iz\cdot\xi} \cdot (i\partial_{\xi_j})^N a(x,\xi)\, d\xi$$

$$= i^N \widecheck{\partial_{\xi_j}^N a}(x,z).$$

The formal calculation above works when $a \in S^{-\infty}$.

For general a, one can still show (using that Fourier transform on S' intertwines multiplication & differentiation) that

$$z_j^N \check{a}(x,z) = i^N \underbrace{\partial_{\xi_j}^N a}\,(x,z).$$

Now since $a \in S^{\ell}(U \times \mathbb{R}^n)$ is a Kohn-Nirenberg symbol we have

$$\partial_{\xi_j}^N a \in \underset{\substack{\uparrow \\ \text{improved by } \langle\xi\rangle^{-N}}}{S^{\ell - N}}(\bar{U} \times \mathbb{R}^n).$$

If N is large, we can then write an honest integral

$$\underbrace{\partial_{\xi_j}^N a}\,(x,z) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iz\cdot\xi} \partial_{\xi_j}^N a(x,\xi)\, d\xi$$

More precisely, if $k + \ell - N < -n$ (i.e. $N > k + \ell + n$) then we can differentiate the above $\int$ k times in z (and any number of times in x) and the integrand is $O(\langle\xi\rangle^k \langle\xi\rangle^{\ell - N})$, $\underline{\text{integrable}}$, which gives $\underbrace{\partial_{\xi_j}^N a} \in C^k(U \times \mathbb{R}^n)$.

Thus $\forall k$ $\exists N$ such that

$$z_j^N \check{a}(x,z) \in C^k(U \times \mathbb{R}^n).$$

Then $(x_j - y_j)^N \mathcal{K}(x,y) \in C^k(U \times U)$

which shows that

$\mathcal{K}$ is C^k on $\{(x,y) \in U \times U \mid x_j \neq y_j\}$

This works $\forall k$, so

$$\mathcal{K} \in C^\infty(\{(x,y) \in U \times U \mid x_j \neq y_j\})$$

as needed.

(This is similar to sing supp $E \subset \{0\}$ for Elliptic Regularity II in §12.2)

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9 From 8 we get that

$\forall a \in S^l(U \times \mathbb{R}^n)$,

$Op(a)$ is <u>pseudolocal</u>:

$\forall u \in \mathcal{E}'(U)$, $\operatorname{sing\,supp}(Op(a)u) \subset$

$\subset \operatorname{sing\,supp} u.$

Indeed, 8 can be reformulated as follows:

$\forall\, \chi_1, \chi_2 \in C^\infty(\bar{U})$ s.t.

$\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$,

the operator $\chi_1 \operatorname{Op}(a) \chi_2 : C_c^\infty(\bar{U}) \to C^\infty(\bar{U})$ is smoothing, i.e. extends to an operator $\mathcal{E}'(U) \to C^\infty(U)$.

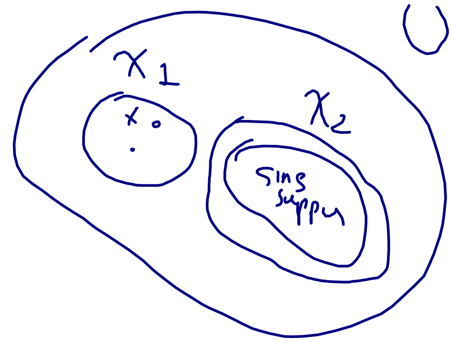
Indeed, the Schwartz kernel of $\chi_1 \operatorname{Op}(a) \chi_2$ is $\chi_1(x)\chi_2(y) K(x,y)$ which is in $C^\infty(U \times U)$ since $\operatorname{supp}(\chi_1(x)\chi_2(y))$ does not intersect the diagonal [see again Pset 5 Problem 1]

Coming back to proof of pseudolocality:

assume $x^0 \in \bar{U} \setminus \operatorname{sing\,supp} u$.

Since $u \in \mathcal{E}'(\bar{U})$, $\operatorname{sing\,supp} u$ is compact.

Fix cutoffs $\chi_1, \chi_2 \in C_c^\infty(U)$ with



- $\chi_1(x^\circ) \neq 0$
- $\chi_2 = 1$ near sing supp u
- $\operatorname{supp} \chi_1 \cap \operatorname{supp} \chi_2 = \emptyset$.

Then write

$$\chi_1 \operatorname{Op}(a) u = \chi_1 \operatorname{Op}(a) \chi_2 u + \chi_1 \operatorname{Op}(a)(1-\chi_2) u \in C_c^\infty(U)$$

Since $\chi_1 \operatorname{Op}(a) \chi_2$ is smoothing

and $(1-\chi_2)u \in C_c^\infty(U)$,

$$\operatorname{Op}(a) : C_c^\infty(U) \to C^\infty(U)$$

<u>Note:</u> pseudolocality also holds for the transpose $\operatorname{Op}(a)^t$

Since the Schwartz kernel of $\operatorname{Op}(a)^t$ is $K(y,x)$, still C^∞ away from the diagonal

§15.3. Elliptic parametrix

Here we show

Thm Assume $P \in \mathrm{Diff}^m(U)$ is elliptic. Then there exists an operator

$$Q: \begin{array}{l} C_c^\infty(U) \to C^\infty(\bar{U}) \text{ and} \\ \mathcal{E}'(\bar{U}) \to \mathcal{D}'(\bar{U}) \text{ such that} \end{array}$$

1. $PQ - I$ is a smoothing operator (in particular, $\forall u \in \mathcal{E}'(\bar{U})$, $PQu - u \in C^\infty(\bar{U})$)
2. Q is pseudolocal, i.e. $\forall u \in \mathcal{E}'(\bar{U})$, $\mathrm{sing\,supp}\, Qu \subset \mathrm{sing\,supp}\, u$.

To prove this, we will put $Q := Op(q)$ for some symbol $q \in S^{-m}(U \times \mathbb{R}^n)$. Note that ② will hold for any q.

To get 1, we compute for any $a \in S^\ell(U \times \mathbb{R}^n)$, the operator $P\,\mathrm{Op}(a)$:

$$\mathrm{Op}(a)\varphi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\,\hat{\varphi}(\xi)\,d\xi$$

$$\left(\text{for } \varphi \in C_c^\infty(U)\right)$$

So for $P = \sum_{|\alpha|\le m} p_\alpha(x) D_x^\alpha$,

$$P\,\mathrm{Op}(a)\varphi(x) =$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \sum_{|\alpha|\le m} p_\alpha(x) D_x^\alpha\left(e^{ix\cdot\xi} a(x,\xi)\right)\hat{\varphi}(\xi)\,d\xi$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \sum_{|\alpha|\le m} e^{ix\cdot\xi} p_\alpha(x)\,(D_x+\xi)^\alpha a(x,\xi)\,\hat{\varphi}(\xi)\,d\xi$$

$$= \mathrm{Op}(P\#a)\,\varphi(x) \quad \text{where}$$

$P\#a \in C^\infty(U \times \mathbb{R}^n)$,

$$P\#a(x,\xi) = \sum_{|\alpha|\le m} p_\alpha(x)(D_x+\xi)^\alpha a(x,\xi),$$

$$(D_x+\xi)^\alpha := (D_{x_1}+\xi_1)^{\alpha_1} \cdots (D_{x_n}+\xi_n)^{\alpha_n}.$$

We can expand out $(D_x+\xi)^\alpha a$:
get a sum of terms of the form
constant. $D_x^\beta \xi^\gamma a$ where $\beta+\gamma=\alpha$

Now, $a \in S^\ell \Rightarrow D_x^\beta \xi^\gamma a \in S^{\ell+|\gamma|}$.

So the leading term is $\beta=\emptyset$, $\gamma=\alpha$, $|\alpha|=m$
giving $\xi^\alpha a \in S^{\ell+m}$,
and the rest is in $S^{\ell+m-1}$.

That is, $a \in S^\ell(U\times\mathbb{R}^n) \Rightarrow$
$\Rightarrow P\# a \in S^{\ell+m}(U\times\mathbb{R}^n)$ and

$$P\# a = p\cdot a + r \quad \text{where}$$

$p=\sigma_m(P)$ (principal symbol: $p(x,\xi)=\sum_{|\alpha|=m} p_\alpha(x)\xi^\alpha$)

and $r \in S^{\ell+m-1}(U\times\mathbb{R}^n)$
(lower order term).

Recall $P\,Op(a) = Op(P\# a)$.

So to set (1), i.e. $PQ - I$ is smoothing

with $Q = Op(q)$, we need

$$\boxed{P\# q = 1 + S^{-\infty}(U \times \mathbb{R}^n)} \quad (*)$$

Since $Op(1) = I$, so

$$PQ - I = Op(P\# q - 1).$$

We first solve $(*)$ with S^{-1} remainder:

- Since P is elliptic, $p = \sigma_m(P)$ is nonzero when $\xi \neq 0$.

And it's a homogeneous polynomial of degree m.

Fix $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi = 1$ near 0

and put $q_0(x,\xi) := \dfrac{1 - \chi(\xi)}{p(x,\xi)}$.

Then $q_0 \in C^\infty(U \times \mathbb{R}^n)$.

Actually, q_0 is a symbol of order $-m$:

$$q_0 \in S^{-m}(U \times \mathbb{R}^n).$$

Indeed, if $|\xi| \gg 1$ so that $\xi \notin \operatorname{supp} \chi$, then $\partial_x^\alpha \partial_\xi^\beta q_0(x,\xi)$ is a linear combination of

$$\frac{\partial_x^{\alpha_1} \partial_\xi^{\beta_1} p(x,\xi) \cdots \partial_x^{\alpha_j} \partial_\xi^{\beta_j} p(x,\xi)}{p(x,\xi)^{j+1}}$$

where $\alpha_1 + \cdots + \alpha_j = \alpha$, $\beta_1 + \cdots + \beta_j = \beta$

Assume $x \in K$ for some compact $K \subset U$.

Then $p \in S^m \Rightarrow$ the numerator is

$$O(\langle \xi \rangle^{jm - |\beta|})$$

and p elliptic $\Rightarrow$ the denominator is $\geq c \langle \xi \rangle^{(j+1)m}$ in absolute value for some $c > 0$.

So $\partial_x^\alpha \partial_\xi^\beta q_0 = O(\langle \xi \rangle^{-m-|\beta|}) \Rightarrow q_0 \in S^{-m}$.

Now, we have ($p \in S^m$, $q_0 \in S^{-m}$)

$$P \# q_0 = pq_0 + S^{-1} = 1 + S^{-1}(U \times \mathbb{R}^n)$$

Since $pq_0 = 1 - \chi(\xi) = 1 + S^{-\infty}$

(as $\chi \in C_c^\infty$)

So we solved (*) modulo S^{-1}.

We next improve to S^{-k} remainder $\forall k \in \mathbb{N}_0$:

$P \# q_0 = 1 - r_1$ where $r_1 \in S^{-1}(U \times \mathbb{R}^n)$.

Define $q_1(x,\xi) = q_0(x,\xi)\, r_1(x,\xi) =$

$$= \frac{1-\chi(\xi)}{p(x,\xi)}\, r_1(x,\xi).$$

Since $q_0 \in S^{-m}$ and $r_1 \in S^{-1}$ we get

$$q_1 \in S^{-m-1}(U \times \mathbb{R}^n).$$

And $P \# q_1 = pq_1 + S^{-2} = r_1 + S^{-2}$

Since $pq_1 = (1-\chi(\xi))\, r_1 = r_1 + S^{-\infty}$

So $P\#(q_0 + q_1) = 1 + S^{1-2}$

i.e. $P\#(q_0 + q_1) = 1 - r_2$

for some $r_2 \in S^{-2}$.

Put $q_2 := q_0 r_2 \in S^{-m-2}$, then

$P\#(q_0 + q_1 + q_2) = 1 + S^{-3}$.

Continuing this process, we construct $q_k \in S^{-m-k}(U \times \mathbb{R}^n)$,

$k = 0, 1, 2, \dots$

Such that $\forall k$,

$P\#(q_0 + q_1 + \dots + q_k) - 1 \in S^{-k-1}(U \times \mathbb{R}^n)$.

How to get $1 + S^{-\infty}$ though?

We use the following

Borel's Theorem

Assume we are given arbitrary

$$q_k \in S^{\ell-k}(U \times \mathbb{R}^n), \quad k = 0, 1, 2, \dots$$

for some $\ell \in \mathbb{R}$. Then

there exists $q \in S^\ell(U \times \mathbb{R}^n)$

which is an asymptotic series

$q \sim \sum_{k=0}^{\infty} q_k$ in the following sense:

$$\forall k, \quad q - q_0 - q_1 - \cdots - q_k \in S^{\ell-k-1}(U \times \mathbb{R}^n).$$

Proof: will give later ... "$\square$"

So we can take $q \sim \sum_{k=0}^{\infty} q_k$

with q_k constructed above,

$$q \in S^{-m}(U \times \mathbb{R}^n).$$

Estimate $P\#q$:

for each $k \in \mathbb{N}_0$,

$$q = q_0 + q_1 + \cdots + q_k + S^{-m-k-1}$$

So $P\#q = P\#(q_0 + q_1 + \cdots + q_k) + S^{-k-1}$

$$= 1 + S^{-k-1}.$$

Since this holds $\forall k$, we have

$$P\#q - 1 \in \bigcap_{k\geq 0} S^{-k-1}(U \times \mathbb{R}^n) = S^{-\infty}(\mathbb{R}^n).$$

So q solves $(*)$ which finishes the proof. $\square$

§15.4. Proof of elliptic regularity

It is enough to work on open subsets of $\mathbb{R}^n$:

if M is a mfld, then enough to show

$$(\operatorname{sing\,supp} u) \cap U_0 \subset \operatorname{sing\,supp} Pu$$

on each domain of a coordinate system

$$\varkappa : U_0 \to V_0$$

and pulling back by $\varkappa$ we reduce to the statement for $\varkappa^{-*}P$ on V_0.

So from now on $U \subset \mathbb{R}^n$ open

$P \in \operatorname{Diff}^m(U)$ elliptic

$u \in \mathcal{D}'(U)$.

Assume that $x_0 \in U$

and $x_0 \notin \operatorname{sing\,supp}(Pu)$.

We need to show $\boxed{x_0 \notin \operatorname{sing\,supp} u.}$

Fix $\chi \in C_c^\infty(U)$ with

$\chi = 1$ near x_0

and define $v := \chi u \in \mathcal{E}'(U)$.

We construct $\tilde{Q}$: $\mathcal{E}'(U) \to \mathcal{D}'(U)$
$C_c^\infty(U) \to C^\infty(U)$

Such that

1. $\tilde{Q}P - I$ is smoothing
2. $\tilde{Q}$ is pseudolocal.

To do this, note that the transpose operator $P^t = \sum_{|\alpha| \le m} D_x^\alpha\, p_\alpha(x) \in \operatorname{Diff}^m(U)$

is elliptic since $\sigma_m(P^t)(x,\xi) = \sigma_m(P)(x,-\xi)$

Let Q be the elliptic parametrix of P^t and put $\tilde{Q} := Q^t$.

Then $\tilde{Q}$ is pseudolocal (since Q was) and

$$\tilde{Q}P - I = Q^t (P^t)^t - I$$

$$= (P^t Q - I)^t \quad (\text{as } (AB)^t = B^t A^t)$$

is smoothing since $P^t Q - I$ was smoothing.

Having constructed $\tilde{Q}$, we write

$$I = \tilde{Q}P + R, \quad R: \mathcal{E}'(U) \to C^\infty(U)$$

Apply to $v \in \mathcal{E}'(U)$, get

$$v = \tilde{Q}Pv + Rv, \quad Rv \in C^\infty(U)$$

So $\operatorname{sing\,supp} v \subset \operatorname{sing\,supp}(\tilde{Q}Pv)$
(since $\tilde{Q}$ is pseudolocal) $\subset \operatorname{sing\,supp}(Pv)$.

And $Pv = P\chi u = \chi P u + [P, \chi]u$.

Now $x_0 \notin \operatorname{sing\,supp} Pu$
& $x_0 \notin \operatorname{supp} [P, \chi]u \Rightarrow$
$\Rightarrow x_0 \notin \operatorname{sing\,supp} Pv \Rightarrow x_0 \notin \operatorname{sing\,supp} v \Rightarrow x_0 \notin \operatorname{sing\,supp} u.$

$\square$

§15.5. Proof of Borel's Thm

Recall we are given $q_k \in S^{\ell-k}(U \times \mathbb{R}^n)$

$k = 0, 1, 2, \dots$

We want to show that there exists $q \in S^{\ell}(U \times \mathbb{R}^n)$

such that $q \sim \sum_{k=0}^{\infty} q_k$.

Fix $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi = 1$ on $B(0,1)$

Take a positive sequence $\varepsilon_k \to 0$ to be fixed later

Put $\boxed{q(x,\xi) = \sum_{k=0}^{\infty} \tilde{q}_k(x,\xi)}$ where

$$\boxed{\tilde{q}_k(x,\xi) = (1 - \chi(\varepsilon_k \xi))\, q_k(x,\xi).}$$

The series above converges pointwise; in fact it is locally finite: if $|\xi| \leq R$ for some R then $\tilde{q}_k(x,\xi) = 0$ when k is large enough depending on R, more precisely when $\varepsilon_k R < 1$.

To make $q \sim \sum_{k=0}^{\infty} q_k$ we need to choose $\varepsilon_k \to 0$ fast enough.

Note that $1 - \chi(\varepsilon \xi) \xrightarrow[\varepsilon \to 0]{} 0$ in $S^1(V \times \mathbb{R}^n)$ (note: not true in S^0!)

i.e. $\forall \alpha, \beta$ we have

$$\sup_\xi |\langle \xi \rangle^{-1+|\beta|} \partial_x^\alpha \partial_\xi^\beta (1 - \chi(\varepsilon \xi))| \xrightarrow[\varepsilon \to 0+]{} 0.$$

Indeed, we may assume that $\alpha = (0, \dots, 0)$

If $\beta = (0,\dots,0)$ then this is

$$\sup_\xi |\langle \xi \rangle^{-1} (1 - \chi(\varepsilon \xi))| \xrightarrow[\varepsilon \to 0+]{} 0$$

Since $|1 - \chi(\varepsilon \xi)| \le C \varepsilon |\xi|$.

For other β this is

$$\sup_\xi |\langle \xi \rangle^{-1+|\beta|} \varepsilon^{|\beta|} (\partial_\xi^\beta \chi)(\varepsilon \xi)| \xrightarrow[\varepsilon \to 0+]{} 0$$

Since on the support of that we have $|\xi| \le \frac{C}{\varepsilon}$.

By Leibniz Rule and since $q_k \in S^{\ell - k}$ we have

$$(1 - \chi(\varepsilon \xi)) q_k(x, \xi) \xrightarrow[\varepsilon \to 0]{} 0 \text{ in } S^{\ell - k + 1}.$$

Fix a family of compact sets

$$K_0 \subset K_1 \subset \cdots \subset U, \quad U = \bigcup_{k=0}^{\infty} K_k.$$

Then we can take ε_k small enough so that $\forall x \in K_k$, $\xi \in \mathbb{R}^n$, $|\alpha|, |\beta| \le k$ we have

$$|\partial_x^\alpha \partial_\xi^\beta \tilde{q}_k(x,\xi)| \le 2^{-k} \cdot \langle \xi \rangle^{\ell-k+1-|\beta|}.$$

Now let's prove that $q \sim \sum_{k=0}^{\infty} q_k$:

We need to estimate $\forall \alpha, \beta, k, \forall$ compact $K \subset U$

$$|\partial_x^\alpha \partial_\xi^\beta (q - q_0 - \cdots - q_k)(x,\xi)| \le C \langle \xi \rangle^{\ell-k-1-|\beta|}, \quad x \in K.$$

Fix N large enough so that

$$|\alpha|, |\beta| \le N, \quad k+1 \le N, \quad K \subset K_N.$$

Then enough to show: $\exists C \, \forall x \in K \, \forall \xi$

$$|\partial_x^\alpha \partial_\xi^\beta (q - q_0 - \cdots - q_N)(x,\xi)| \le C \langle \xi \rangle^{\ell-k-1-|\beta|}$$

because $q_{k+1} + \cdots + q_N \in S^{\ell-k-1}$

(as $q_{k+1} \in S^{\ell-k-1}, \ldots, q_N \in S^{\ell-N}$)

Now, we write

$$q - q_0 - \cdots - q_N = \sum_{j=0}^{N} (\tilde{q}_j - q_j) + \sum_{j=N+1}^{\infty} \tilde{q}_j .$$

The first term is actually in $S^{-\infty}$

since $\tilde{q}_j - q_j = -\chi(\varepsilon_j \xi) q_j$ and $\varepsilon_j > 0$

supp χ compact

As for the second one: $\forall x \in K \subset K_N$, $\forall \xi$

$$\left| \partial_x^\alpha \partial_\xi^\beta \left(\sum_{j=N+1}^{\infty} \tilde{q}_j (x,\xi) \right) \right| \leq$$

$$\leq \sum_{j=N+1}^{\infty} \left| \partial_x^\alpha \partial_\xi^\beta \tilde{q}_j (x,\xi) \right|$$

$$\leq \sum_{j=N+1}^{\infty} 2^{-j} \langle \xi \rangle^{\ell - j + 1 - |\beta|} \leq C \langle \xi \rangle^{\ell - N - |\beta|}$$

$$\leq C \langle \xi \rangle^{\ell - k - 1 - |\beta|} .$$

which finishes the proof

that $q \sim \sum_{k=0}^{\infty} q_k .$ $\square$