18.118, SPRING 2022, PROBLEM SET 5

Review/useful information:

- Bowen metric for a map: if (X, d) is a metric space and $\varphi : X \to X$ is continuous, then
$$d_{n,\varphi}(x, y) = \max\{d(\varphi^j(x), \varphi^j(y)) \mid 0 \le j < n\}.$$
- Topological entropy of a map: $h_{\text{top}}(\varphi) = \lim_{\varepsilon \to 0+} h_\varepsilon(\varphi)$ where
$$h_\varepsilon(\varphi) = \lim_{n\to\infty} \frac{\log D_\varphi(\varepsilon, n)}{n}$$
and $D_\varphi(\varepsilon, n)$ is the smallest number of sets which cover X and have $d_{n,\varphi}$-diameter no more than ε.
- Bowen metric for a flow: if $\varphi^t : X \to X$ is a one-parameter continuous group of continuous maps then for $T \ge 0$
$$d_{T,\varphi}(x, y) = \sup\{d(\varphi^t(x), \varphi^t(y)) \mid 0 \le t \le T\}.$$
The topological entropy of a flow is defined similarly to that of a map.
- Entropy of a partition: if $\xi = (A_\ell)_{\ell=1}^m$ and μ is a probability measure then
$$H_\mu(\xi) := -\sum_{\ell=1}^m \mu(A_\ell) \log \mu(A_\ell).$$
- Refined partition by a map φ: if $\xi = (A_\ell)_{\ell=1}^m$ then
$$\xi^{(n)} := \bigvee_{j=0}^{n-1} \varphi^{-j}(\xi) = \left\{ \bigcap_{j=0}^{n-1} \varphi^{-j}(A_{w_j}) \,\middle|\, w_0, \dots, w_{n-1} \in \{1, \dots, m\} \right\}.$$
- Entropy of a map φ with respect to a measure μ:
$$h_\mu(\varphi) = \sup\{h_\mu(\varphi, \xi) \mid \xi \text{ a finite partition}\},$$
$$h_\mu(\varphi, \xi) = \lim_{n\to\infty} \frac{H_\mu(\xi^{(n)})}{n}.$$

1. Let $\varphi^t : X \to X$ be a continuous flow on a compact metric space X. Show that the topological entropy of the flow φ is equal to the topological entropy of its time-one map φ^1.

2. Let $\varphi : X \to X$ be a diffeomorphism of a compact manifold X. Show that $h_{\text{top}}(\varphi)$ is finite. (Hint: you can bound it in terms of the Lipschitz constant of φ and the dimension $m = \dim X$. You might want to use the fact that if μ is a smooth volume measure on X, then $\mu(B(x,r)) \geq C^{-1}r^m$ for all $x \in X$ and $0 < r < 1$.)

3. Let $\varphi : X \to X$ be an Anosov map (we assume that $\dim X > 0$ which implies that $\dim E_u, \dim E_s > 0$). Show that $h_{\mathrm{top}}(\varphi) > 0$. You may use the following quantitative version of the Stable/Unstable Manifold Theorem: there exists $\lambda \in (0,1)$ such that for $\varepsilon > 0$ small enough and all $n \geq 0$, if $d_{n,\varphi}(x,y) \leq \varepsilon$ then $d(y, W^s(x)) \leq \lambda^n$, where $W^s(x)$ is the stable manifold centered at x. (It is actually possible to recover this statement from the version that we studied in class but you need not do this here.)

4. (Optional) Consider the map on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$

$$\varphi : \mathbb{S}^1 \to \mathbb{S}^1, \quad \varphi(x) = 3x \bmod \mathbb{Z}.$$

Let $X \subset [0,1]$ be the mid-third Cantor set. We think of it as a subset of $\mathbb{S}^1$. Show that $\varphi(X) \subset X$ and compute the topological entropy of the restriction $\varphi|_X$.

5. Let $\varphi : X \to X$ be an Anosov diffeomorphism. As in Problem 3 of the previous problemset, denote by Z_n the set of periodic points of φ of period n. Show that for each $\delta > 0$

$$|Z_n| = \mathcal{O}(e^{(h_{\mathrm{top}}(\varphi)+\delta)n}) \quad \text{as} \quad n \to \infty.$$

6. Assume that $\varphi : X \to X$ is a map preserving a probability measure μ. Show that for each $k \geq 1$ we have $h_\mu(\varphi^k) = k h_\mu(\varphi)$ and, if φ is invertible, then $h_\mu(\varphi^{-1}) = h_\mu(\varphi)$. (Hint: for the first part, show that $k h_\mu(\varphi, \xi) = h_\mu(\varphi^k, \widetilde{\xi})$ for an appropriate choice of a partition $\widetilde{\xi}$.)

7. (Optional) Let $X = \mathbb{R}/\mathbb{Z}$ and $\varphi(x) = 2x \bmod \mathbb{Z}$. Find a sequence μ_k of φ-invariant probability measures on X which converges weakly to some measure μ and $h_{\mu_k}(\varphi) = 0$ for all k, yet $h_\mu(\varphi) > 0$. This shows that entropy is not a continuous function on the space of measures with weak convergence. (Hint: try to take each μ_k to be supported on finitely many points, whose number grows with k.)

8. (Optional) Consider the map φ and the set X from Problem 4. Fix $0 < b < 1$. Let μ_b be the *Bernoulli convolution* which is a probability measure supported on X with the following property: for each word $\mathbf{w} = w_1 \dots w_n$, where $w_1, \dots, w_n \in \{0, 2\}$, if

$$I_{\mathbf{w}} := \Big(\sum_{j=1}^{n} w_j 3^{-j}\Big) + [0, 3^{-n}]$$

is one of the intervals featured in the construction of the Cantor set X, then

$$\mu_b(I_{\mathbf{w}}) = b^{k_{\mathbf{w}}}(1-b)^{n-k_{\mathbf{w}}} \quad \text{where} \quad k_{\mathbf{w}} = \#\big\{j \in \{1, \dots, n\} \mid w_j = 0\big\}.$$

(Such μ_b exists, is unique, and is φ-invariant, where the latter follows from the fact that $\varphi^{-1}(I_{\mathbf{w}}) \cap X = I_{0\mathbf{w}} \sqcup I_{2\mathbf{w}}$. You do not need to check this in your solution. One can think of μ_b as the distribution of the random variable $\sum_{j=1}^{\infty} \omega_j 3^{-j}$ where ω_j are i.i.d. Bernoulli random variables with $\mathbb{P}(\omega_j = 0) = b$, $\mathbb{P}(\omega_j = 2) = 1 - b$. Taking $b = \frac{1}{2}$ gives the standard Cantor measure on the mid-third Cantor set.) Compute $h_\mu(\varphi)$. You may use what we did in lecture: $h_\mu(\varphi) = h_\mu(\varphi, \xi)$ if ξ is a partition such that $\max\{\mathrm{diam}(A) \mid A \in \xi^{(n)}\} \to 0$ as $n \to \infty$.