18.118, SPRING 2022, PROBLEM SET 2

Review/useful information:

- Contact form on a $2d-1$-dimensional manifold X: a 1-form α on X such that $\alpha \wedge (d\alpha)^{d-1} \neq 0$ everywhere.
- Reeb vector field for a contact 1-form: the unique $V \in C^\infty(X; TX)$ such that $\alpha(V) = 1$ and $\iota_V d\alpha = 0$.
- Anosov flow $\varphi^t = e^{tV} : X \to X$: V has no fixed points, X is compact, and for each $x \in X$ we have the flow/stable/unstable decomposition $T_xX = E_0(x) \oplus E_s(x) \oplus E_u(x)$ with $E_0(x) = \mathbb{R}V(x)$ and there exist $C, \theta > 0$ such that for all $(x, v) \in TX$

$$|d\varphi^t(x)v| \leq Ce^{-\theta|t|}|v|, \quad \begin{cases} t \geq 0, & v \in E_s(x), \\ t \leq 0, & v \in E_u(x). \end{cases}$$

1. Let X be a manifold and $\varphi : X \to X$ be a diffeomorphism. Assume that $x_0 \in X$ is a periodic point for φ, that is $\varphi^r(x_0) = x_0$ for some $r \geq 1$. Show that the periodic orbit $\gamma = \{\varphi^j(x_0) \mid 0 \leq j < r\}$ is a hyperbolic set for φ if and only if the linear homomorphism $d\varphi^r(x_0) : T_{x_0}X \to T_{x_0}X$ has no eigenvalues on the unit circle in $\mathbb{C}$.

2. Let $\widetilde{X}$ be a compact manifold, $\widetilde{\varphi} : \widetilde{X} \to \widetilde{X}$ be a diffeomorphism, and $\widetilde{\mu}$ be a $\widetilde{\varphi}$-invariant probability measure on $\widetilde{X}$. Let (X, φ^t) be the suspension of $\widetilde{\varphi}$ with roof function $\tau \equiv 1$; here X is glued from the cylinder $\widetilde{X}_x \times [0,1]_s$. Show that φ^t cannot be mixing with respect to the measure $\mu = \widetilde{\mu} \times ds$, that is there exist $f, g \in L^2(X)$ such that

$$\int_X f(g \circ \varphi^t)\, d\mu \not\to \Big(\int_X f\, d\mu\Big)\Big(\int_X g\, d\mu\Big) \quad \text{as} \quad t \to \infty.$$

3. (Optional) This exercise shows that a suspension flow over a compact manifold is never a contact flow. Let $\widetilde{X}$ be a compact $2d$-dimensional manifold and put $X := \widetilde{X} \times (0,1)_s$. Show that the vector field ∂_s cannot be the Reeb vector field of any contact 1-form on X. (Hint: assume that α is a contact form with Reeb vector field ∂_s. Show that $\alpha = ds + j^*\beta$ where $j : X \to \widetilde{X}$ is the projection map and β is a 1-form on $\widetilde{X}$. Then show that such α cannot be a contact form, by integrating $(d\beta)^{\wedge d}$ over $\widetilde{X}$.)

4. Assume that $\varphi^t = e^{tV} : X \to X$ is an Anosov flow and the manifold X is connected. Assume that $f \in C^1(X)$ is a φ^t-invariant function, that is $f \circ \varphi^t = f$ for all $t \in \mathbb{R}$. Show that f is constant. (Hint: fix $(x, v) \in TX$. Using φ^t-invariance of f for $t \geq 0$, show that $df(x)v = 0$ if $v \in E_s(x)$. Arguing similarly with $t \leq 0$, show that $df(x)v = 0$

if $v \in E_u(x)$. Show also that $df(x)V(x) = 0$, and conclude that f is constant. This is a baby version of *Hopf's argument* that we will study soon.)

5. Assume that X is a compact manifold and $\varphi^t = e^{tV}$ is an Anosov flow on X, which is also a contact flow, i.e. V is the Reeb vector field of some contact 1-form α. Show that the kernel of α is given by $E_u \oplus E_s$. (Hint: use that α is φ^t-invariant to show that $\alpha(x)v = 0$ for $v \in E_u$ and for $v \in E_s$.)