§7. Properties of hyperbolic systems

§7.1. Continuity of stable/unstable spaces

Let us focus on the case of maps here. The case of flows is handled similarly.

Let X be a manifold and $\varphi: X \to X$ a diffeomorphism.

Let $K \subset X$ be a compact φ-invariant set such that φ is hyperbolic on K (see §4.1)

Recall that this gives a stable/unstable decomposition

$$x \in K \Rightarrow T_x X = E_u(x) \oplus E_s(x).$$

Here we will show that $E_u(x)$, $E_s(x)$ depend continuously on $x \in K$.

(With more work one can show Hölder continuity: C^ε for some possibly small $\varepsilon > 0$)

To make sense of continuity of E_u, E_s, let $\dim E_u(x) = d_u$, $\dim E_s(x) = d_s$ $\forall x \in K$

(we assumed that the dimension is constant)

and define the <u>Grassmanian bundles</u>

$G_u = \{(x, E) \mid x \in M,\ E \subset T_x M$ is a d_u-dimensional subspace$\}$

and similarly for G_s (taking d_s-dim. subspaces)

Then G_u, G_s are smooth manifolds and the projections π: $\begin{matrix} G_u \\ \downarrow \\ M \end{matrix}$, $\begin{matrix} G_s \\ \downarrow \\ M \end{matrix}$,

$\pi(x, E) = x$, are fibrations

(In local coordinates, this reduces to showing that the set of d_u-dimensional subspaces of $\mathbb{R}^d$ can be made into a manifold.)

Now we can think of

$$x \in K \longmapsto E_u(x) \subset T_x M$$

$$x \in K \longmapsto E_s(x) \subset T_x M$$

as maps $E_u : K \to G_u$

$$E_s : K \to G_s.$$

<u>Thm</u> The maps E_u, E_s are continuous.

<u>Rmk</u> Same is true for hyperbolic flows, with the same proof.

<u>Proof</u> We just consider the case of E_s.

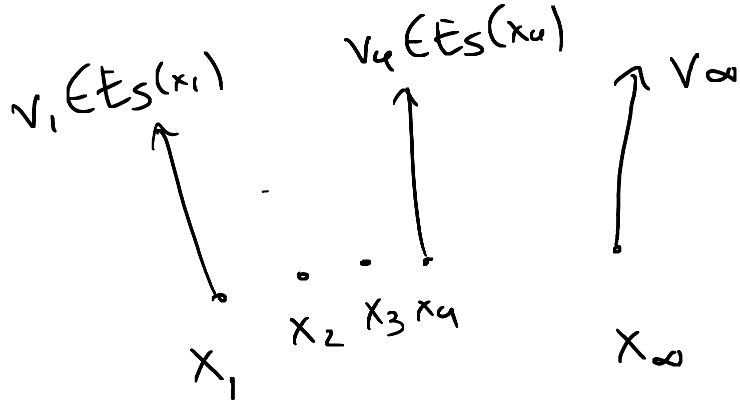
It suffices to show sequential continuity: if $x_k \in K$,

$$x_k \xrightarrow[k\to\infty]{} x_\infty \in K$$

then $E_s(x_k) \to E_s(x_\infty)$ in the topology of G_u.

To show that $E_s(x_k) \to E_s(x_\infty)$ it suffices to prove the following: if we take arbitrary $\boxed{v_k \in E_s(x_k)}$ such that $|v_k| = 1$ and $\boxed{v_k \xrightarrow[k\to\infty]{} v_\infty} \in T_{x_\infty} M$ (in the topology of TM) then $\boxed{v_\infty \in E_s(x_\infty).}$ $(*)$

<u>Picture:</u>



(Think of a basic case: $d = \dim X = 2$, $d_s = 1$ each 1D subspace of $\mathbb{R}^2$ is characterized by its slope : $G_s(X) \sim$ a circle ... and our statement is that $\text{slope}(E_s(x_k)) \xrightarrow[k\to\infty]{} \text{slope}(E_s(x_\infty))$)

Now, to show $(*)$,
note that $v_k \in E_s(x_k)$ implies that
$\forall n \geq 0$, $\boxed{|d\varphi^n(x_k)v_k| \leq C\lambda^n}$
where $C>0$, $\lambda \in (0,1)$ are from the definition of a hyperbolic map.
Here C, λ are independent of n, k.
So we can pass to the limit as $k \to \infty$,
obtaining for each n
$$\boxed{|d\varphi^n(x_\infty)v_\infty| \leq C\lambda^n}.$$
But this implies that $v_\infty \in E_s(x_\infty)$:
indeed, we can write $v_\infty = v_u + v_s$
where $v_u \in E_u(x_\infty)$, $v_s \in E_s(x_\infty)$.
We have $|d\varphi^n(x_\infty)v_s| \xrightarrow[n\to\infty]{} 0$ and
$v_u \neq 0 \Rightarrow |d\varphi^n(x_\infty)v_u| \xrightarrow[n\to\infty]{} \infty$.
So $|d\varphi^n(x_\infty)(v_u+v_s)| \xrightarrow[n\to\infty]{} 0$ means that
$v_u = 0$, i.e. $v_\infty \in E_s(x_\infty)$. $\square$

§7.2. Adapted metrics

Assume again that $\varphi: X \circlearrowleft$ is a diffeomorphism. Recall that a φ-invariant set K is hyperbolic if there exists a $d\varphi$-invariant splitting

$$T_x X = E_u(x) \oplus E_s(x), \quad x \in K$$

where $\exists\, C > 0,\ \lambda \in (0,1)$:

$$\forall x \in K, \ v \in T_x X$$

$$|d\varphi^n(x) v| \le C \lambda^{|n|}, \quad \begin{cases} n \geq 0, & v \in E_s(x) \\ n \le 0, & v \in E_u(x). \end{cases}$$

The above holds for any choice of Riemannian metric on X, with the constant C depending on the metric.

I.e. $d\varphi^n$ is eventually contracting

eventually means for large enough $|n|$

Here we show that one can choose a metric to make $d\varphi^n$ <u>immediately</u> contracting,
(i.e. for $|n|=1$)

That is make $C=1$ above:

<u>Thm</u> Assume φ is hyperbolic on K.
Then there exist C^∞ Riemannian metrics $|\cdot|_u$, $|\cdot|_s$ on X and $\tilde{\lambda} \in (0,1)$ such that
$\forall x \in K$, $v \in T_x X$

$$|d\varphi(x)v|_s \leq \tilde{\lambda}|v|_s, \text{ if } v \in E_s(x)$$

$$|d\varphi^{-1}(x)v|_u \leq \tilde{\lambda}|v|_u, \text{ if } v \in E_u(x).$$

<u>Proof</u> We just construct $|\cdot|_s$
(for $|\cdot|_u$, replace φ by φ^{-1})

Fix any Riem. metric $|\cdot|$ on X
and any $\tilde{\lambda}$: $\lambda < \tilde{\lambda} < 1$
where λ is in the eventually contracting property.

Put $\forall\ x \in X,\ v \in T_x X$

$$|v|_s^2 = \sum_{n=0}^{N-1} \tilde{\lambda}^{-2n} |d\varphi^n(x) v|^2$$

where $N \gg 1$ will be chosen later.
(N does not depend on x or v)

Then $|\cdot|_s$ is a C^∞ Riemannian metric on X (owing to the squares in the formula above)

Now, let $v \in E_s(x)$ $(x \in K)$

and compute

$$|d\varphi(x)v|_s^2 = \sum_{n=0}^{N-1} \tilde{\lambda}^{-2n} |d\varphi^n(\varphi(x))\, d\varphi(x) v|_s^2$$

$$= \sum_{n=0}^{N-1} \tilde{\lambda}^{-2n} |d\varphi^{n+1}(x) v|_s^2 = \sum_{n=1}^{N} \tilde{\lambda}^{2-2n} |d\varphi^n(x) v|_s^2$$

$$= \tilde{\lambda}^2 \left(|v|_s^2 - |v|^2 + \tilde{\lambda}^{-2N} |d\varphi^N(x) v|^2 \right)$$

$\leq \tilde{\lambda}^2 |v|_s^2$ as needed, provided that

$$|d\varphi^N(x) v| \leq \tilde{\lambda}^N |v|.$$

The latter can be arranged by taking N large enough: by hyperbolicity of φ, and since $v \in E_s(x)$, we have

$$|d\varphi^N(x)v| \leq C\lambda^N |v|$$

for some fixed constant C & all N.

It remains to take N large enough so that $C\lambda^N \leq \tilde{\lambda}^N$ (recall $\tilde{\lambda} > \lambda$). $\square$

<u>Remark</u>: A similar statement holds for hyperbolic <u>flows</u>, with $\exists\, \tilde{\theta} > 0$: $\forall\, x \in K$, $v \in T_x X$, t

$$|d\varphi^t(x)v|_s \leq e^{-\tilde{\theta}|t|} \cdot |v|_s, \text{ if } t \geq 0, \ v \in E_s(x)$$

$$|d\varphi^t(x)v|_u \leq e^{-\tilde{\theta}|t|} \cdot |v|_u, \text{ if } t \leq 0, \ v \in E_u(x)$$

See e.g. [D, Lemma 4.7]

§7.3. Stable and unstable cones

We work here mostly in the setting of maps, generalizing to flows at the end.

Let $\varphi: X \circlearrowleft$ be a diffeomorphism and K be a compact φ-invariant set.

Recall from §4 that φ is hyperbolic on K if we have the decomposition into stable/unstable spaces

$$T_x X = E_u(x) \oplus E_s(x), \qquad x \in K$$

This definition in general not easy to check since we'd have to construct the subspaces E_u, E_s which are typically not smooth in x (so it's hopeless to try some simple formula: the cases we studied so far, i.e. cat maps & geodesic flows on hyperbolic surfaces, are very special)

In this section we give a more robust characterization of hyperbolicity, using stable/unstable cone fields.

Denote ($K \subset X$ compact)

$$T_K X \setminus 0 = \{(x,v) \in TX \mid x \in K,\ v \neq 0\}$$

We say a subset $\mathcal{C} \subset T_K X \setminus 0$ is a cone field (over K) if

$$(x,v) \in \mathcal{C} \Rightarrow (x, sv) \in \mathcal{C} \quad \forall s > 0$$

Note: if we fix some continuous norm on the fibers of TX, then a cone field $\mathcal{C}$ is characterized by its intersection with $S_K X = \{(x,v) \in T_K X : |v| = 1\}$ and $S_K X$ is compact.

We say that a cone field $\mathcal{C}$ is closed if it's closed as a subset of $T_X K \setminus 0$

(same as $\mathcal{C} \cap S_K X$ closed)

Similarly define open cone fields.

If $\varphi: X \to X$ is a diffeo. preserving K, then for each cone field $\mathcal{C} \subset T_K X \setminus 0$ we can define the propagated cone field

$$\varphi_* \mathcal{C} \subset T_K X \setminus 0,$$

$$\varphi_* \mathcal{C} = \left\{ (\varphi(x), d\varphi(x)v) \;\middle|\; \begin{matrix} x \in K \\ (x,v) \in \mathcal{C} \end{matrix} \right\}$$

i.e. if we put for $x \in K$

$$\mathcal{C}_x = \{ v \in T_x X \setminus 0 \mid (x,v) \in \mathcal{C} \}$$

then $(\varphi_* \mathcal{C})_{\varphi(x)} = d\varphi(x)\, \mathcal{C}_x$

Similarly we can define $\varphi_*^{-1} \mathcal{C}$.

In what follows, assume that $\dim X = d$ and write $\boxed{d = d_u + d_s}$ for some d_u, d_s (stable/unstable dimensions)

We are now ready to formulate an equivalent definition of hyperbolicity

Defn We say φ is cone-hyperbolic on K if there exist closed cone fields

$$\boxed{\mathcal{C}^u, \mathcal{C}^s \subset T_K X \setminus 0}$$

and a continuous norm $|\cdot|$ on the fibers of $T_K X$ such that:

1. $\mathcal{C}^u \cap \mathcal{C}^s = \emptyset$ (transversality)
2. $\forall x \in K$, $\mathcal{C}^u_x$ contains a d_u- dimensional space & (modulo 0) $\mathcal{C}^s_x$ contains a d_s- dimensional space
3. $\varphi_* \mathcal{C}^u \subset \operatorname{int} \mathcal{C}^u$ ← interior (in the topology of $T_K X \setminus 0$) and $\varphi_*^{-1} \mathcal{C}^s \subset \operatorname{int} \mathcal{C}^s$
4. We have: $\exists\ \alpha > 1$:

$\forall (x,v) \in \mathcal{C}^u$, $|d\varphi(x)v| > \alpha |v|$ (expansion on unstable cones) and

$\forall (x,v) \in \mathcal{C}^s$, $|d\varphi^{-1}(x)v| > \alpha|v|$ (backwards expansion on stable cones)

18.118
7-14

Example: $X = \mathbb{R}^2_{x,y}$, $K = \{(0,0)\}$

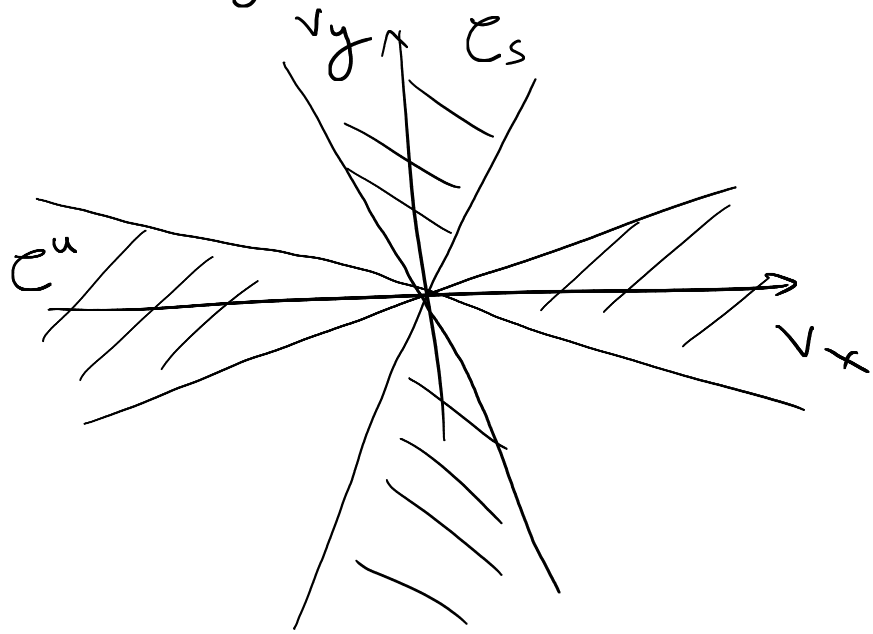
$\varphi(x,y) = (2x, \frac{y}{2})$.

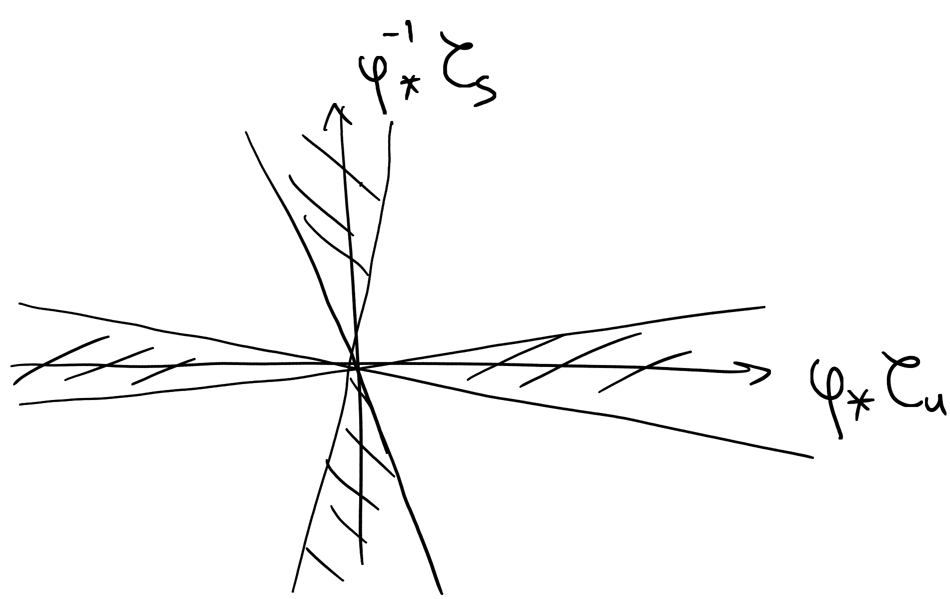
This is cone-hyperbolic with e.g.

$\mathcal{C}^u = \{(0, v_x, v_y) : |v_y| \le \frac{1}{2}|v_x|, \ v_x \neq 0\}$

$\mathcal{C}^s = \{(0, v_x, v_y) : |v_x| \le \frac{1}{2}|v_y|, \ v_y \neq 0\}$

Picture:





We compute

$$\varphi_* \mathcal{C}^u = \{(0, v_x, v_y) : |v_y| \leq \tfrac{1}{8}|v_x|,\ v_x \neq 0\}$$

$$\varphi_*^{-1} \mathcal{C}^s = \{(0, v_x, v_y) : |v_x| \leq \tfrac{1}{8}|v_y|,\ v_y \neq 0\}$$

So we see that properties 1 - 3 in the definition above hold.

If $(0, v_x, v_y) \in \mathcal{C}^u$, then $|v_y| \leq \frac{1}{2}|v_x|$ and

$d\varphi(0)(v_x, v_y) = (2v_x, \frac{v_y}{2})$, so, using the norm $|(v_x, v_y)| = \max(|v_x|, |v_y|)$

$$|d\varphi(0)(v_x, v_y)| \geq 2|v_x| \geq 2|(v_x, v_y)| > |(v_x, v_y)|$$

Similarly if $(0, v_x, v_y) \in \mathcal{C}^s$, then

$$|d\varphi^{-1}(0)(v_x, v_y)| > |(v_x, v_y)|.$$

So the property 4 holds as well.

We now state

Thm φ is hyperbolic on K $\Leftrightarrow$
$\Leftrightarrow$ φ is cone-hyperbolic on K.

Before proving it, we give a

Corollary Assume $\varphi : X \circlearrowleft$ is an Anosov map. Then $\exists \varepsilon > 0$:
$\forall$ $\psi : X \circlearrowleft$ a diffeomorphism with $\|\varphi - \psi\|_{C^1} < \varepsilon$,
ψ is also an Anosov map.
(Anosov condition is stable under C^1-perturbations)

Proof Take the stable/unstable cones $\mathcal{C}^u, \mathcal{C}^s$ and the norm $|\cdot|$ for the cone-hyperbolicity of φ.
Then for $\|\varphi - \psi\|_{C^1}$ small enough, they also give cone-hyperbolicity of ψ.
(1-4 are open conditions on φ) $\square$

Proof of Thm

HYPERBOLICITY $\Rightarrow$ CONE-HYPERBOLICITY:

Assume φ is hyperbolic on K.
Let E_u, E_s be the unstable/stable spaces & fix adapted metrics
$|\cdot|_s, |\cdot|_u$ (see §7.2)

Define the norm $|\cdot|$ on the fibers of $T_K X$
by $\boxed{|v| := \max(|v_u|_u, |v_s|_s)}$
where $\boxed{v = v_u + v_s,}$ $v_u \in E_u(x)$, $v_s \in E_s(x)$
This is continuous since E_u, E_s depend continuously on x.

Now, define the cones $\mathcal{C}^u, \mathcal{C}^s$ by

$$\mathcal{C}^u = \{(x,v) : x \in K, \ |v_s| \le \tfrac{1}{2}|v_u|, \ v_u \ne 0\}$$

$$\mathcal{C}^s = \{(x,v) : x \in K, \ |v_u| \le \tfrac{1}{2}|v_s|, \ v_s \ne 0\}$$

We have $\mathcal{C}^u \cap \mathcal{C}^s = \emptyset$ and

$$E_u(x) \setminus 0 \subset \mathcal{C}^u_x$$
$$E_s(x) \setminus 0 \subset \mathcal{C}^s_x$$

So properties 1 - 2 in the definition of cone-hyperbolicity hold.

We now check that 3 & 4 hold for $\mathcal{C}^u$ (similar argument for $\mathcal{C}^s$).

Let $(x, v) \in \mathcal{C}^u$. Then write $v = v_u + v_s$, $v_u \in E_u(x)$, $v_s \in E_s(x)$.

We have $\varphi_*(x, v) = (\varphi(x), \tilde{v})$ where

$\tilde{v} = \tilde{v}_u + \tilde{v}_s$, $\tilde{v}_u = d\varphi(x) v_u$, $\tilde{v}_s = d\varphi(x) v_s$.

By the properties of adapted metrics, $\exists \lambda \in (0,1)$:

$$|\tilde{v}_u| \geq \lambda^{-1} |v_u|, \quad |\tilde{v}_s| \leq \lambda |v_s|$$

Since $v \in \mathcal{C}^u_x$, we have $|v_s| \leq \frac{1}{2} |v_u|$, so $|v| = \max(|v_u|, |v_s|) = |v_u|$. Now

$$|\tilde{v}_s| \leq \lambda |v_s| \leq \frac{\lambda}{2} |v_u| \leq \frac{\lambda^2}{2} |\tilde{v}_u| \quad \text{and} \quad \frac{\lambda^2}{2} < \frac{1}{2}$$

So $\varphi_*(x, v) \in$ interior of $\mathcal{C}_u$ giving property 3

Finally, $|\tilde{v}| \geq |\tilde{v}_u| \geq \lambda^{-1} |v_u| = \lambda^{-1} |v| > |v|$ which gives property 4.

CONE-HYPERBOLICITY $\Rightarrow$ HYPERBOLICITY

Assume now we ~~take~~ the cones $\mathcal{C}^u, \mathcal{C}^s$ & the norm $|\cdot|$ on the fibers of $T_K X$ such that the definition of cone-hyperbolicity holds.

Fix $x \in K$. By (2) we know that $\exists$ subspaces $V_u(x), V_s(x) \subset T_x X$:

$\dim V_u(x) = d_u$, $\dim V_s(x) = d_s$,

$V_u(x) \setminus 0 \subset \mathcal{C}^u_x$, $V_s(x) \setminus 0 \subset \mathcal{C}^s_x$.

For $n \geq 0$, define the <u>propagated cone</u>

$$\mathcal{C}^{u,n}_x := (\varphi^n_* \mathcal{C}^u)_x = d\varphi^n(\varphi^{-n}(x))\, \mathcal{C}^u_{\varphi^{-n}(x)}.$$

Each $\mathcal{C}^{u,n}_x$ is a closed conic subset of $T_x X \setminus 0$ and

$$\mathcal{C}^{u,n+1}_x \subset \mathcal{C}^{u,n}_x,$$

as follows from the fact that

$$\varphi_* \mathcal{C}^u \subset \mathcal{C}^u \text{ and thus}$$

$$\varphi^{n+1}_* \mathcal{C}^u = \varphi^n_* \varphi_* \mathcal{C}_u \subset \varphi^n_* \mathcal{C}_u.$$

18.118
7-20

So, we have a nested family of closed cones in $T_xX \setminus 0$,

$$\mathcal{C}^u_x = \mathcal{C}^{u,0}_x \supset \mathcal{C}^{u,1}_x \supset \mathcal{C}^{u,2}_x \supset \cdots$$

Define $E_u(x) \subset T_xX$ by

$$\boxed{E_u(x) = \{0\} \cup \left(\bigcap_n \mathcal{C}^{u,n}_x \right),}$$

i.e. a nonzero v lies in $E_u(x)$ if it lies in <u>any</u> forward iteration under φ_* of the unstable cone $\mathcal{C}^u$.

We claim that

(*) $E_u(x)$ is a d_u-dimensional subspace of T_xX

The proof of that has <u>2 steps</u>:

1. $\exists$ a d_u-dimensional subspace $W_u \subset T_xX$ s.t. $W_u \subset E_u(x)$.

To see this, let

$$G^n := \{ W \subset T_xX \ \ d_u\text{-dimensional subspace s.t. } W \setminus 0 \subset \mathcal{C}^{u,n}_x \}.$$

(with the topology from the Grassmanian of d_u-dim subspaces of T_xX)

Since each $\mathcal{C}_X^{u,n}$ is a closed conical subset of $T_x X \setminus 0$, each G^n is closed (in the Grassmanian) and thus compact.

We also have $\mathcal{C}_x^{u,n+1} \subset \mathcal{C}_x^{u,n} \Rightarrow$

$$\Rightarrow G^{n+1} \subset G^n.$$

And $G^n \neq \emptyset$ since $d\varphi^n(\varphi^{-n}(x)) V_u(\varphi^{-n}(x)) \in G^n$.

By the nested-set criterion for compactness, we have

$$\bigcap_n G^n \neq \emptyset.$$

It remains to take

$$W_u \in \bigcap_n G^n, \text{ then}$$

$\forall n$, $W_u \setminus 0 \subset \mathcal{C}_X^{u,n}$, so

$W_u \subset E_u(x)$ as needed.

(2.) We now claim that

$$E_u(x) \subset W_u.$$

This shows that $E_u(x) = W_u$ & gives $(*)$.

It suffices to prove that

$(**)$ $v', v'' \in E_u(x),\ v' - v'' \in V_s(x) \Rightarrow v' = v''.$

Indeed, $T_x X = W_u \oplus V_s(x)$

Since $\dim W_u = d_u$, $\dim V_s(x) = d_s$,

$d_u + d_s = d$, and

$W_u \cap V_s(x) = 0$ as

$W_u \setminus 0 \subset \mathcal{C}^u_x$, $V_s(x) \setminus 0 \subset \mathcal{C}^s_x$

and $\mathcal{C}^u_x \cap \mathcal{C}^s_x = \emptyset$ (by property (1))

So for each $v \in E_u(x)$ we have

$v = v_1 + v_2$ where $v_1 \in W_u$, $v_2 \in V_s(x)$

but $v_2 = v - v_1$; $v, v_1 \in E_u(x)$

and $v_2 \in V_s(x)$, so by $(**)$

we get $v_2 = 0 \Rightarrow v = v_1 \Rightarrow v \in W_u$,

So $E_u(x) \subset W_u$ as needed.

We now show (**).

Assume that

$v', v'' \in E_u(x)$, $w := v' - v'' \in V_s(x)$.

For each $n \geq 0$, we have

$v' \in \mathcal{C}_x^{u,n} \cup 0 \Rightarrow$

$\Rightarrow d\varphi^{-(n+1)}(x)\, v' \in \mathcal{C}^u_{\varphi^{-(n+1)}(x)} \Rightarrow$ (by property 4)

$\Rightarrow |d\varphi^{-(n+1)}(x) v'| = |d\varphi^{-1}(\varphi^{-n}(x)) d\varphi^{-n}(x) v'|$

$\leq \alpha^{-1} |d\varphi^{-n}(x) v'|$ where $\alpha > 1$ is fixed

$\Rightarrow$ by induction,

$|d\varphi^{-n}(x) v'| \leq \alpha^{-n} |v'|$ and similarly

$|d\varphi^{-n}(x) v''| \leq \alpha^{-n} |v''|$

So $|d\varphi^{-n}(x) w| \leq |d\varphi^{-n}(x) v'| + |d\varphi^{-n}(x) v''|$

and $|d\varphi^{-n}(x) w| \xrightarrow[n \to \infty]{} 0$.

But if $w \neq 0$ then $w \in \mathcal{C}^s_x$ (as $V_s(x) \setminus 0 \subset \mathcal{C}^s_x$)

and thus (by property 3)

$d\varphi^{-n}(x) w \in \mathcal{C}^s_{\varphi^{-n}(x)} \cup 0 \quad \forall n \geq 0$

So by property 4

$$|d\varphi^{-(n+1)}(x)w| \geq |d\varphi^{-n}(x)w|.$$

So the only way for $|d\varphi^{-n}(x)w| \xrightarrow[n\to\infty]{} 0$ is that $w=0$, which gives $(**)$.

We have shown $(*)$ and constructed $\forall x \in K$ a d_u-dimensional subspace $E_u(x) \subset T_x K$.

From the construction we have that E_u is $d\varphi$-invariant:

$d\varphi(x) E_u(x) = E_u(\varphi(x))$ and

$E_u(x) \subset \mathcal{C}^u_x \cup 0, \ \forall x.$

By property 4 we have: $\exists \alpha > 1$:

$$|d\varphi(x)v| \geq \alpha |v| \quad \forall v \in E_u(x).$$

which gives $|d\varphi^{-1}(x)v| \leq \lambda |v|$ where

$$\lambda = \alpha^{-1}$$

So we have exponential contraction backwards in time on E_u:

$$|d\varphi^{-n}(x)v| \leq \lambda^n |v| \quad \forall v \in E_u(x), \ \forall n \geq 0$$

We can similarly construct the stable space $E_s(x) \subset \mathcal{C}^s_x \cup 0$ with $|d\varphi^n(x)v| \leq \lambda^n |v| \ \forall v \in E_s(x) \ \forall n \geq 0$

And $T_x X = E_u(x) \oplus E_s(x)$

since $\dim E_u(x) + \dim E_s(x) = d_u + d_s = d$

and $E_u(x) \cap E_s(x) = 0$

since $\mathcal{C}^u \cap \mathcal{C}^s = \emptyset$.

This gives the stable/unstable decomposition and shows that φ is hyperbolic on K. $\square$

We finally state cone hyperbolicity for flows:

Thm Let $\varphi^t = e^{tV} : X \circlearrowleft$ be a flow and $K \subset X$ a compact φ^t-invariant set. Then φ^t is hyperbolic on K iff it is cone-hyperbolic, defined as follows: there exists a continuous norm $|\cdot|$ on the fibers of $T_K X$ and cone fields $\mathcal{C}^u, \mathcal{C}^s \subset T_K X \setminus 0$ s.t.:

① If $v_u \in \mathcal{C}^u_x \cup 0$, $v_s \in \mathcal{C}^s_x \cup 0$ and $v_u + v_s \in \mathbb{R} V(x)$ then $v_u = v_s = 0$

② $\forall x \in K$ $\exists$ subspaces $V_u(x), V_s(x) \subset T_x X$ such that $\dim V_u(x) = d_u$, $\dim V_s(x) = d_s$, d_u, d_s fixed, $d_u + d_s + 1 = d = \dim X$ and $V_u(x) \setminus 0 \subset \mathcal{C}^u_x$, $V_s(x) \setminus 0 \subset \mathcal{C}^s_x$

③ $\varphi^1_*(\mathcal{C}^u) \subset \operatorname{int} \mathcal{C}^u$, $\varphi^{-1}_*(\mathcal{C}^s) \subset \operatorname{int} \mathcal{C}^s$.

④ $\exists$ $\alpha > 1$: $\forall x \in K$, $v \in T_x X$

$v \in \mathcal{C}^u_x \Rightarrow |d\varphi^1(x) v| > \alpha |v|$

$v \in \mathcal{C}^s_x \Rightarrow |d\varphi^{-1}(x) v| > \alpha |v|$.

§7.4. Negatively curved surfaces

We now give an application of cone hyperbolicity of §7.3 to the geodesic flow on a negatively curved surface.

- (M,g) compact Riemannian surface of Gauss curvature $K \in C^\infty(M)$
- $\varphi^t = e^{tV}$ the geodesic flow, $\varphi^t : X \to X$ where $X = SM$.

We will show

<u>Thm</u> If $K < 0$ everywhere

Then φ^t is an Anosov flow.

To show the Thm, we first describe the differential of the geodesic flow φ^t using Jacobi fields from Riemannian geometry.

(for an approach not using Jacobi fields, see e.g. [D, §5.1])

If $\gamma : \mathbb{R} \to M$ is a curve and $Y(s) \in T_{\gamma(s)}M$ is a vector field along γ, then denote by

$$D_s Y(s) \in T_{\gamma(s)} M$$

the (Levi-Civita) covariant derivative of $Y(s)$ along γ

(if Y extends to a vector field on M then $D_s Y(t) = \nabla_{\dot\gamma(s)} Y(\gamma(s))$)

To study tangent vectors to SM, can look at <u>paths</u> on SM. Assume that $\tilde{\gamma} : \mathbb{R} \to SM$. Then we can write

$$\tilde{\gamma}(s) = (\gamma(s), Y(s))$$

where $\gamma(s) \in M$ and

$$Y(s) \in S_{\gamma(s)}M \subset T_{\gamma(s)}M.$$

So, γ is a path on M and Y is a vector field along γ.

Now, $\dot{\tilde{\gamma}}(s) \in T_{\tilde{\gamma}(s)}(SM)$ is determined by

$$\dot{\gamma}(s) \in T_{\gamma(s)}M \quad \text{and}$$

$$D_s Y(s) \in T_{\gamma(s)}M$$

18.118
7-30

Moreover, since $\langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle_g = 1$
we can differentiate in s to get

$$\boxed{\langle D_s \dot{\gamma}(s), \dot{\gamma}(s) \rangle_g = 0.}$$

Assume M is oriented
and for each $v \in S_x M$
define $v_\perp \in S_x M$ by rotating v ccw
by $\pi/2$: i.e. $\langle v, v_\perp \rangle_g = 0$
and $v, v_\perp$ is positively oriented.

Then $\tilde{\dot{\gamma}}(0)$ is determined by 3 numbers

$$\xi_1 = \langle \dot{\gamma}(0), v \rangle_g$$

$$\xi_2 = \langle \dot{\gamma}(0), v_\perp \rangle_g$$

$$\xi_3 = \langle D_s \dot{\gamma}(0), v_\perp \rangle_g$$

where $v = \dot{\gamma}(0)$.

This defines for each $(x,v) \in SM$ the linear isomorphism

$$T_{(x,v)}SM \longrightarrow \mathbb{R}^3$$

$$\xi \longmapsto (\xi_1, \xi_2, \xi_3)$$

where for any path

$\tilde{\gamma}(s) = (\gamma(s), \underline{V}(s)) \in SM,$

$\gamma(0) = 0, \quad \underline{V}(0) = v, \quad \dot{\tilde{\gamma}}(0) = \xi,$

we have

$\xi_1 = \langle \dot{\gamma}(0), v \rangle_g, \quad \xi_2 = \langle \dot{\gamma}(0), v_\perp \rangle,$
$\xi_3 = \langle D_s \underline{V}(0), v_\perp \rangle.$

Note that $\dot{\gamma}(0) = d\pi(x,v)\xi$

where $\pi : SM \to M, \quad \pi(x,v) = x.$

and $\xi_3 = 0 \Longleftrightarrow$ going along ξ gives an infinitesimally parallel vector field.
Also, $\xi_1(\xi) = \alpha(\xi)$ where α is the usual contact form.

<u>Example:</u> $M = \mathbb{R}^2$ with the Euclidean metric.

$S\mathbb{R}^2$ parametrized by (x, y, θ) with the point (x,y) and the tangent vector $(\cos\theta, \sin\theta)$

Each $\xi \in T_{(x,y,\theta)} S\mathbb{H}^2$ can be written as $(\xi_x, \xi_y, \xi_\theta)$

and we have

$$\xi_1 = \xi_x \cos\theta + \xi_y \sin\theta$$

$$\xi_2 = -\xi_x \sin\theta + \xi_y \cos\theta$$

$$\xi_3 = \xi_\theta .$$

Indeed, for a path $\tilde{\gamma}(s) = (x(s), y(s), \theta(s))$ we have $\gamma(s) = (x(s), y(s))$ and

$$Y(s) = (\cos\theta(s), \sin\theta(s)),$$

$$D_s Y(s) = (\partial_s \cos\theta(s), \partial_s \sin\theta(s))$$

Now, we describe the differential of the geodesic flow,

$\varphi^t : SM \to SM$.

Assume that $(x,v) \in SM$ and $(\gamma(t), \dot{\gamma}(t)) = \varphi^t(x,v)$ where $\gamma : \mathbb{R} \to M$ is the geodesic such that $\gamma(0) = x$, $\dot{\gamma}(0) = v$.

Let $\xi \in T_{(x,v)} SM$ and

$\xi(t) = d\varphi^t(x,v)\, \xi \in T_{\varphi^t(x,v)} SM$

Define $\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$ accordingly.

Lemma. The functions $\xi_j(t)$ satisfy the system of ODEs

$$\begin{cases} \partial_t \xi_1(t) = 0 \\ \partial_t \xi_2(t) = \xi_3(t) \\ \partial_t \xi_3(t) = -\underbrace{K(\gamma(t))}_{\text{Gauss curvature!}} \xi_2(t) \end{cases}$$

Proof Take some curve

$\tilde{\gamma}(s) = (\gamma(s), Y(s)) \in SM$, $s \in \mathbb{R}$

such that $\tilde{\gamma}(0) = (x, v)$ and

$\partial_s \tilde{\gamma}(0) = \xi$.

Define the curve $\gamma(t,s)$ by

$$(\gamma(t,s), \partial_t \gamma(t,s)) = \varphi^t(\tilde{\gamma}(s))$$

(i.e. the geodesic starting at $\tilde{\gamma}(s)$)

Then $\forall s$, $t \mapsto \gamma(t,s)$ is a geodesic on M.

We have

$$d\varphi^t(x,v)\xi = \xi(t) = \partial_s|_{s=0}\, \varphi^t(\tilde{\delta}(s))$$

Thus

$$\xi_1(t) = \langle \partial_s|_{s=0}\gamma(t,s),\ \partial_t|_{s=0}\gamma(t,s)\rangle_g$$

$$\xi_2(t) = \langle \partial_s|_{s=0}\gamma(t,s),\ \partial_t|_{s=0}\gamma(t,s)_\perp\rangle_g$$

$$\xi_3(t) = \langle D_s|_{s=0}\partial_t\gamma(t,s),\ \partial_t|_{s=0}\gamma(t,s)_\perp\rangle_g.$$

Note that $\partial_t|_{s=0}\gamma(t,s) = \dot{\gamma}(t)$

and $\partial_t|_{s=0}\gamma(t,s) = \dot{\gamma}(t)_\perp$

and, since γ is a geodesic,

$D_t\dot{\gamma}(t) = 0$ and $D_t\dot{\gamma}(t)_\perp = 0.$

So we set

$$\dot{\xi}_1(t) = \langle D_t\partial_s\gamma(t,s),\ \dot{\gamma}(t)\rangle_g|_{s=0}$$

$$\dot{\xi}_2(t) = \langle D_t\partial_s\gamma(t,s),\ \dot{\gamma}(t)_\perp\rangle_g|_{s=0}$$

$$\dot{\xi}_3(t) = \langle D_tD_s\partial_t\gamma(t,s),\ \dot{\gamma}(t)_\perp\rangle_g|_{s=0}$$

But also

$$D_t \partial_s \gamma(t,s) = D_s \partial_t \gamma(t,s), \text{ so}$$

$$\dot{\xi}_1(t) = \langle D_s \partial_t \gamma, \partial_t \gamma \rangle|_{s=0}$$

$$= \tfrac{1}{2} \partial_s \langle \partial_t \gamma, \partial_t \gamma \rangle|_{s=0} = 0$$

since $|\partial_t \gamma| = 1$,

$$\dot{\xi}_2(t) = \langle D_s \partial_t \gamma, \partial_t \gamma_\perp \rangle|_{s=0} = \xi_3(t),$$

$$\dot{\xi}_3(t) = \langle D_t^2 \partial_s \gamma, \partial_t \gamma_\perp \rangle|_{s=0}.$$

Now we use <u>Jacobi's Equation</u>:

since $t \mapsto \gamma(t,s)$ is a geodesic $\forall s$,

$$D_t^2 \partial_s \gamma = -R(\partial_s \gamma, \partial_t \gamma)\partial_t \gamma$$

Riemann curvature tensor,

$$\text{so } \dot{\xi}_3(t) = -\langle R(\partial_s \gamma, \partial_t \gamma)\partial_t \gamma, \partial_t \gamma_\perp \rangle|_{s=0}$$

$$= -K(\gamma(t)) \langle \partial_s \gamma, \partial_t \gamma_\perp \rangle|_{s=0}$$

$$= -K(\gamma(t))\, \xi_2(t). \qquad \square$$

We now use the Lemma to construct stable/unstable cones.

Since $\dot{\xi}_1 = 0$, we can just construct these inside the 2-D subspaces $\{\xi_1 = 0\} \subset T_{(x,v)}SM$ which are actually just $\ker\alpha$ where α is the contact form.

Note that $\ker\alpha$ is φ^t-invariant.

Define first the cones $(\xi \neq 0)$

$$\mathcal{C}^{u,0}(x,v) = \{\xi_1 = 0,\ \xi_2 \cdot \xi_3 \geq 0\}$$

$$\mathcal{C}^{s,0}(x,v) = \{\xi_1 = 0,\ \xi_2 \cdot \xi_3 \leq 0\}.$$

Note that

$$d\varphi^t(\mathcal{C}^{u,0}(x,v)) \subset \mathcal{C}^{u,0}(\varphi^t(x,v)):$$

indeed, $\dot{\xi}_2 = \xi_3$, $\dot{\xi}_3 = -K(t)\,\xi_2$,

so $(\xi_2\xi_3)^{\cdot} = \xi_3^2 - K(t)\,\xi_2^2 \geq 0$ if $K < 0$.

However, $d\varphi^t$ is not expanding on $\mathcal{C}^{u,0}$. So we have to make a small adjustment to get the right cones.
We are now ready to give

<u>Proof of Thm (hyperbolicity of the geodesic flow)</u>

Let $\varepsilon > 0$ be a small constant to be fixed later.

For $(x,v) \in SM$, recall that we are looking at $\ker\alpha(x,v) = \{\xi_1 = 0\}$.
Define the <u>norm</u> $|\cdot|$ on $\ker\alpha(x,v)$ by

$$|\xi| = \sqrt{\xi_2^2 + \xi_3^2}.$$

Also, define the function

$$\Theta : \ker\alpha \setminus \{0\} \to \mathbb{R} \quad \text{by}$$

$$\Theta(\xi) = \frac{\xi_2 \xi_3}{|\xi|^2} = \frac{\xi_2 \xi_3}{\xi_2^2 + \xi_3^2}.$$

Note that Θ is homogeneous of degree 0.

Define the unstable/stable cones

$\mathcal{C}^u, \mathcal{C}^s \subset \ker\alpha \setminus 0$ with

$$\mathcal{C}^u_{(x,v)} = \{\xi \in \ker\alpha \setminus 0 \mid \Theta(\xi) \geq \varepsilon\}$$

$$\mathcal{C}^s_{(x,v)} = \{\xi \in \ker\alpha \setminus 0 \mid \Theta(\xi) \leq -\varepsilon\}$$

Note that $\mathcal{C}^u, \mathcal{C}^s$ are closed cone fields,

① $\mathcal{C}^u \cap \mathcal{C}^s = \emptyset$, and

② $\forall (x,v)$, $\mathcal{C}^u_{(x,v)}$ contains a 1D subspace, for example it contains the line $\{\xi_2 = \xi_3\}$ (assuming we took $\varepsilon \leq \frac{1}{2}$)

and similarly, $\mathcal{C}^s_{(x,v)}$ contains a line, e.g. $\{\xi_2 = -\xi_3\}$.

We claim next that

③ $\forall t \geq 0$,

$$d\varphi^t(x,v)\, \mathcal{C}^u_{(x,v)} \subset \mathcal{C}^u_{\varphi^t(x,v)},$$

$$d\varphi^{-t}(x,v)\, \mathcal{C}^s_{(x,v)} \subset \mathcal{C}^s_{\varphi^{-t}(x,v)}.$$

Let us show that
$$d\varphi^t(\mathcal{C}^u_{(x,v)}) \subset \mathcal{C}^u_{\varphi^t(x,v)},$$
with the case of $\mathcal{C}^s$ handled similarly.

Let $\xi \in \mathcal{C}^u_{(x,v)} \subset \ker \alpha(x,v) \subset T_{(x,v)}SM$.

Define $\xi(t) = d\varphi^t(x,v)\xi$

and look at $\xi_1(t), \xi_2(t), \xi_3(t)$.

By the Lemma, with $K(t) := K(\gamma(t))$
$$\dot{\xi}_1(t) = 0, \quad \dot{\xi}_2(t) = \xi_3(t), \quad \dot{\xi}_3(t) = -K(t)\,\xi_2(t)$$

So $\xi_1(t) = 0 \quad \forall t \quad (\text{as } \xi_1(0) = 0)$

and, denoting
$$R(t) = |\xi(t)|^2 = \xi_2(t)^2 + \xi_3(t)^2$$
$$\Theta(t) = \Theta(\xi(t)) = \frac{\xi_2(t)\,\xi_3(t)}{\xi_2(t)^2 + \xi_3(t)^2},$$

we compute
$$\begin{aligned}\dot{R}(t) &= 2\big(\xi_2(t)\,\dot{\xi}_2(t) + \xi_3(t)\,\dot{\xi}_3(t)\big)\\ &= 2\,\xi_2(t)\,\xi_3(t)\big(1 - K(t)\big)\\ &= 2R(t)\,\Theta(t)\big(1 - K(t)\big)\end{aligned}$$

$$\dot{\Theta}(t) = \frac{\partial_t(\xi_2(t)\xi_3(t))}{R(t)} - \frac{\Theta(t)\dot{R}(t)}{R(t)}$$

$$= \frac{\xi_3(t)^2 - K(t)\,\xi_2(t)^2}{R(t)} - 2\Theta(t)^2(1-K(t))$$

Since $\mathcal{C}^u = \{\Theta \geq \varepsilon\}$, to show ③
we need to check that $\forall t$,

$$\Theta(t) = \varepsilon \Rightarrow \dot{\Theta}(t) > 0. \qquad (*)$$

We have $K(t) > 0$, so
the first part of $\dot{\Theta}(t)$ is > 0,
but the second part could be a problem.

Fix constants $0 < K_0 < K_1$ such that

$$K_0 \leq -K(x) \leq K_1 \quad \forall x \in M,$$

then we bound for $\Theta(t) = \varepsilon$,

$$\dot{\Theta}(t) \geq \min(1, K_0) - 2\varepsilon^2(1+K_1).$$

Fix now $0 < \varepsilon < \sqrt{\frac{\min(1,K_0)}{2(1+K_1)}}$,
then we get $(*)$ and thus ③.

Finally, we claim that $\exists\, \nu > 0$:

(4) $\xi \in \mathcal{C}^u_{(x,v)} \Rightarrow \forall t \geq 0,$

$$|d\varphi^t(x,v)\xi| \geq e^{\nu t}|\xi| \quad \text{and}$$

$\xi \in \mathcal{C}^s_{(x,v)} \Rightarrow \forall t \geq 0,$

$$|d\varphi^{-t}(x,v)\xi| \geq e^{\nu t}|\xi|.$$

We again only prove the 1st statement.

If $\xi \in \mathcal{C}^u_{(x,v)}$, then by (3)

we have $\xi(t) \in \mathcal{C}^u_{\varphi^t(x,v)}$ $\forall t \geq 0$.

So it suffices to show that $\exists\, \nu > 0$:

$$\dot{R}(t) \geq 2\nu R(t), \quad \forall t \geq 0$$

(as $R(t) = |\xi(t)|^2$ and by Gronwall, we have $R(t) \geq e^{2\nu t} R(0)$...)

Recall that

$$\dot{R}(t) = 2R(t)\,\Theta(t)\,(1 - K(t)) \underset{\text{as } K(t)<0}{\geq} 2R(t)\,\Theta(t)$$

$$\geq 2\varepsilon R(t), \text{ as } \Theta(t) \geq \varepsilon \ \forall t \geq 0$$

(as $\xi(t) \in \mathcal{C}^u$)

This gives (4) with $\nu := \varepsilon$.

Now 1 - 4
and the Thm in §7.3
imply that φ^t is
cone-hyperbolic (on the entire SM)
and thus hyperbolic. $\square$