§5. Geodesic flows

Here we study some basic facts about geodesic flows on Riemannian manifolds.
In particular, we see that they are <u>contact flows</u>.

§5.1. Contact flows

Assume that X is a <u>compact manifold</u> (without boundary) of odd dimension $2d-1$.

Denote by $\Omega^k = \Lambda^k T^* X$ the bundle of differential k-forms on X.

<u>Defn</u> A 1-form

$\alpha \in C^\infty(X; \Omega^1)$ is called

a <u>contact form</u> on X, if

the $2d-1$ form

$$dVol_\alpha := \alpha \wedge \underbrace{d\alpha \wedge \dots \wedge d\alpha}_{d-1 \text{ times}}$$

is nonvanishing.

In this case X is <u>oriented</u>

(by requiring $dVol_\alpha > 0$)

and we have a probability measure

μ_α on X defined by

$$\int_X f \, d\mu_\alpha = \frac{\int_X f \, dVol_\alpha}{\int_X dVol_\alpha}$$

μ_α is called the <u>Liouville measure</u>

The Reeb vector field

$V \in C^\infty(X; TX)$ for a contact form is the unique vector field such that

- $\alpha(V) \equiv 1$
- $d\alpha(V, W) = 0$ for any vector field W.

This vector field exists & is unique since (using that $dVol_\alpha(x) \neq 0$) we can check that $\forall x \in X$, the space $\operatorname{Ker} d\alpha(x) = \{V \in T_x X \mid \forall W \in T_x X \; d\alpha(x)(V, W) = 0\}$ is 1-dimensional and transversal to $\ker \alpha(x) = \{V \in T_x X \mid \alpha(x)(V) = 0\}$.

(indeed, we can check first that $dVol_\alpha(x) \neq 0$ gives $\operatorname{Ker} \alpha \cap \ker d\alpha = \{0\}$ but also $\operatorname{Ker} d\alpha \neq \{0\}$ since $d\alpha$ is an antisymmetric bilinear form on the odd-dimensional space $T_x X$.)

- The contact flow is $\varphi^t = e^{tV}$, the flow of the Reeb vector field.

- The Liouville measure μ "=" $d\mathrm{Vol}_\alpha$ is φ^t-invariant: by Cartan's magic f-la
$$\mathcal{L}_V \alpha = d(\iota_V \alpha) + \iota_V d\alpha = d(1) + 0 = 0$$
So $\mathcal{L}_V(\alpha \wedge d\alpha \wedge \dots \wedge d\alpha) = 0$.

Example: $X = \mathbb{T}^3 = \mathbb{R}^3 / 2\pi\mathbb{Z}^3$

Take coordinates (x, y, θ) on X where $x, y, \theta \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$

Put $\boxed{\alpha = \cos\theta \, dx + \sin\theta \, dy}$

Then $d\alpha = \sin\theta \, dx \wedge d\theta - \cos\theta \, dy \wedge d\theta$

and $\alpha \wedge d\alpha = -dx \wedge dy \wedge d\theta = d\mathrm{Vol}_\alpha$

Reeb vector field:
$$V = \cos\theta \, \partial_x + \sin\theta \, \partial_y.$$

§5.2. Geodesic flows

Let (M, g) be a (compact) Riemannian manifold, $\dim M = d$.

A C^∞ curve $\gamma : \mathbb{R} \to M$ is called a <u>geodesic</u> if

$D_t \dot{\gamma}(t) = 0$ where $\dot{\gamma}(t) \in T_{\gamma(t)} M$ is the velocity vector of γ ($\dot{\gamma} = \partial_t \gamma$) and D_t is the Levi-Civita covariant derivative of vector fields along γ. (see e.g. 18.965 ...)

<u>In coordinates:</u> $g = \sum_{j,k} g_{jk}(x) dx^j dx^k$

If $\gamma(t) = (x^1(t), \dots, x^d(t))$ then the geodesic equation is

$$\underbrace{\ddot{x}^\ell(t)}_{\partial_t^2 x^\ell(t)} + \sum_{j,k} \Gamma^\ell_{jk}(\gamma(t))\, \dot{x}^j(t)\, \dot{x}^k(t) = 0$$

where $\Gamma^{\ell}_{jk}(x)$ are the (coordinate dependent) Christoffel symbols of the Levi-Civita connection:

$$\Gamma^{\ell}_{jk}(x) = \frac{1}{2}\sum_{m} g^{\ell m}(x)\left(\frac{\partial g_{mj}}{\partial x^k}(x) + \frac{\partial g_{mk}}{\partial x^j}(x) - \frac{\partial g_{jk}}{\partial x^m}(x)\right)$$

where the matrix $(g^{\ell m}(x))_{\ell,m=1}^{d}$ is the inverse of the matrix $(g_{jk}(x))_{j,k=1}^{d}$.

To define the geodesic flow on M, consider the sphere bundle (unit tangent bundle)

$$X = SM = \{(x,\xi) : x \in M,\ \xi \in T_xM,\ |\xi|_{g(x)} = 1\}$$

Note: X is a (compact) $2d-1$-dimensional manifold.

Defn. The <u>geodesic flow</u>

$\varphi^t : X \to X$, $t \in \mathbb{R}$, $X = SM$,

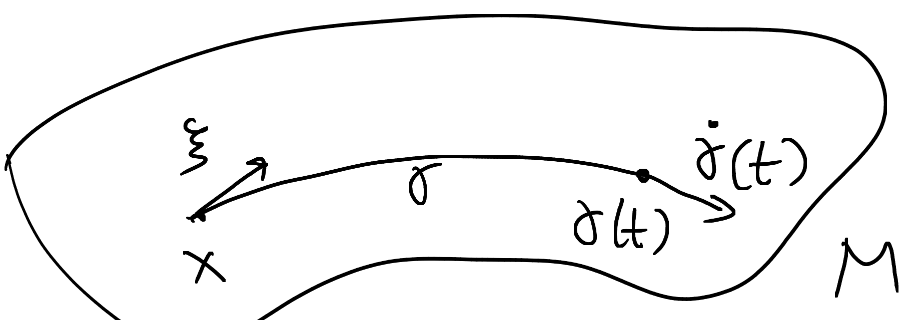
is defined as follows:

for each $(x, \xi) \in SM$, if

$\gamma : \mathbb{R} \to M$ is the (unique)

geodesic with $\gamma(0) = x$, $\dot\gamma(0) = \xi$

then $\varphi^t(x, \xi) = (\gamma(t), \dot\gamma(t))$.



We can write $\varphi^t = e^{tV}$ where

$V \in C^\infty(X; TX)$, $X = SM$, is

the <u>generator</u> of the geodesic flow.

In coordinates:

if $\varphi^t(x,\xi) = (\gamma(t), \dot\gamma(t))$ and

$\gamma(t) = (x^1(t), \dots, x^d(t))$,

$\dot\gamma(t) = \xi^1(t)\partial_{x^1} + \dots + \xi^d(t)\partial_{x^d}$

then x^ℓ, ξ^ℓ satisfy the geodesic eq^n:

$$\begin{cases} \dot x^\ell = \xi^\ell \\ \dot\xi^\ell = -\sum_{j,k=1}^{d} \Gamma^\ell_{jk}(\gamma(t))\,\xi^j\xi^k \end{cases}$$

Thus in the coordinates (x,ξ)

on $SM \subset TM$, we have

$$\boxed{V = \sum_\ell \xi^\ell \partial_{x^\ell} - \sum_{j,k,\ell} \Gamma^\ell_{jk}(x)\,\xi^j\xi^k\,\partial_{\xi^\ell}.}$$

Example: $M = \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, coordinates (x,y), $x,y \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$

$g = dx^2 + dy^2$, $X = SM = \mathbb{T}^3_{x,y,\theta}$, where

the tangent vector $\xi = \cos\theta\cdot\partial_x + \sin\theta\cdot\partial_y$.

$\Gamma^\ell_{jk} = 0 \Rightarrow V = \cos\theta\cdot\partial_x + \sin\theta\cdot\partial_y$

the vector field from the example in §5.1.

Now, we show that the geodesic flow is a contact flow.

Define the 1-form $\alpha \in C^\infty(X; \Omega^1)$ as follows: for $(x,\xi) \in X = SM$ and $v \in T_{(x,\xi)}X$, put

$$\alpha(x,\xi)(v) = g(x)(\xi, d\pi(x,\xi)\,v)$$

where $\pi : SM \to M$ is the projection map, $d\pi(x,\xi) : T_{(x,\xi)}SM \to T_x M$.

<u>In local coordinates (x,ξ) on SM:</u>

$$\boxed{\alpha = \sum_{j,k=1}^{d} g_{jk}(x)\,\xi^j\,dx^k}$$

<u>**Thm**</u> α is a contact form on $X = SM$ and its Reeb vector field is V, the generator of the geodesic flow.

Proof

(1.) We first show that $\alpha(V)=1$. This follows from a direct computation:

$$(\alpha, V) = \sum_{j,k} g_{jk}(x)\,\xi^j \xi^k = |\xi|^2_{g(x)} = 1$$

since $(x,\xi) \in SM$, so $|\xi|_{g(x)} = 1$.

(2.) We next show that $\iota_V d\alpha = 0$ where $\iota_V d\alpha$ is the 1-form defined by: for all $w \in T_{(x,\xi)}X$

$$\boxed{\iota_V d\alpha\,(x,\xi)(w) = d\alpha(x,\xi)(V,w)}$$

We compute

$$d\alpha = \sum_{j,k} g_{jk}(x)\, d\xi^j \wedge dx^k + \sum_{j,k,\ell} \partial_{x^\ell} g_{jk}(x)\,\xi^j dx^\ell \wedge dx^k.$$

$$= \qquad \beta_1 \qquad + \beta_2.$$

We compute

$$\iota_V \beta_2 = \sum_{j,k,\ell} \partial_{x^\ell} g_{jk}(x)\left(\xi^j \xi^\ell dx^k - \xi^j \xi^k dx^\ell\right)$$

$$= \sum_{j,k,\ell} \left(\partial_{x^\ell} g_{jk}(x) - \partial_{x^k} g_{j\ell}(x)\right)\xi^j \xi^\ell dx^k$$

and

$$\iota_V \beta_1 = -\sum_{j,k,p,q} g_{jk}(x)\,\Gamma^j_{pq}(x)\,\xi^p\xi^q\,dx^k - \sum_{j,k} g_{j\ell}(x)\,\xi^\ell\,d\xi^j .$$

From the formula for Γ^j_{pq}

we have $\sum_j g_{jk}(x)\,\Gamma^j_{pq}(x) =$

$$= \frac{1}{2}\left(\partial_{x^p} g_{kq}(x) + \partial_{x^q} g_{kp}(x) - \partial_{x^k} g_{pq}(x)\right),$$

So we can write (using that $g_{jk} = g_{kj}$)

$$\iota_V d\alpha = -\frac{1}{2}\sum_{j,k,\ell} \partial_{x^k} g_{j\ell}(x)\,\xi^j\xi^\ell\,dx^k - \sum_{j,\ell} g_{j\ell}(x)\,\xi^\ell\,d\xi^j .$$

Now this is $= 0$ as a 1-form on $X = SM$

since $\iota_V d\alpha = -\frac{1}{2}\,d\left(\sum_{j,\ell} g_{j\ell}(x)\,\xi^j\xi^\ell\right)$

and $\sum_{j,\ell} g_{j\ell}(x)\,\xi^j\xi^\ell = |\xi|^2_{g(x)} = 1$ on X.

3) We showed that

$$\alpha(V) = 1$$

$$\iota_V d\alpha = 0.$$

It remains to prove that α is a contact form, i.e.

$$\alpha \wedge \underbrace{d\alpha \wedge \cdots \wedge d\alpha}_{d-1 \text{ times}} \neq 0 \text{ everywhere.}$$

For that, fix $x_0 \in M$ and a system of coordinates on M such that

$$g_{jk}(x_0) = \begin{cases} 1, & \text{if } j=k \\ 0, & \text{otherwise.} \end{cases}$$

Also, fix $\boxed{\xi_0 = \partial_{x^1}.}$

Then take the tangent vectors $v_1, \dots, v_d \in T_{(x_0,\xi_0)}X$ given by $v_j = \partial_{x^j} - \frac{\partial_{x^j} g_{11}(x_0)}{g_{11}(x_0)} \partial_{\xi^1}$ and the tangent vectors $\partial_{\xi_2}, \dots, \partial_{\xi_d} \in T_{(x_0,\xi_0)}X$ where we recall that $X = SM$ is defined by the equation $\sum_{k,l=1}^{d} g_{kl}(x)\xi^k\xi^l = 1.$

We compute

$$\alpha(x_0,\xi_0) = dx_1,$$

$$d\alpha(x_0,\xi_0) = \sum_{j=1}^{d} d\xi^j \wedge dx^j + (\text{sth. with } dx^\ell \wedge dx^k).$$

Now, when we apply

$$\alpha \wedge \underbrace{d\alpha \wedge \dots \wedge d\alpha}_{d-1 \text{ times}} (x_0,\xi_0) \text{ to the}$$

vectors $v_1, \dots, v_d, \partial_{\xi_2}, \dots, \partial_{\xi_d} \in T_{(x_0,\xi_0)}X$,

using the identities

$$\alpha(v_1) = 1, \quad \alpha(v_j) = \alpha(\partial_{\xi_j}) = 0 \text{ for } j \geq 2$$

$$d\alpha(\partial_{\xi_j}, \partial_{\xi_\ell}) = 0,$$

$$d\alpha(\partial_{\xi_j}, v_\ell) = \begin{cases} 1, & \text{if } j=\ell \\ 0, & \text{otherwise} \end{cases}$$

With a bit of work we get

$$(\alpha \wedge d\alpha \wedge \dots \wedge d\alpha)(x_0,\xi_0)(v_1,\dots,v_d,\partial_{\xi_2},\dots,\partial_{\xi_d}) =$$

$$= \underset{\substack{\uparrow \\ \text{depends only on } d}}{\pm} (d-1)!$$

. So, $\alpha \wedge d\alpha \wedge \dots \wedge d\alpha \neq 0$.

$\square$

<u>Remark</u> The proof above shows that the Liouville measure μ for the geodesic flow as a contact flow is, up to a constant, just the natural volume measure on $X = SM$:

$$\int_X f\,d\mu = c \int_M \left(\int_{S_x M} f(x,\xi)\, dS_x(\xi) \right) d\mathrm{Vol}_g(x)$$

for some $c > 0$ (so that $\mu(X) = 1$)

where dS_x is the standard measure on the sphere $S_x M = \{\xi \in T_x M : |\xi|_{g(x)} = 1\}$ induced by the metric $g(x)$

and $d\mathrm{Vol}_g$ is the Riemannian volume measure of the metric g.

For more on the basics of geodesic flows, see e.g. Paternain, "Geodesic Flows", §§1.1 – 1.3.

§5.3. Results on geodesic & contact flows

Here we state a few fundamental results. We will prove some of these at least in special cases.

Others, such as exponential mixing, are outside of the scope of this course.

We start with

Thm. [Anosov 1960s] Assume that (M, g) is a compact Riemannian manifold of negative sectional curvature, i.e.

(R = Riemann curvature tensor)

$$\langle R(A,B)B, A\rangle_g < 0$$

for all linearly independent A, B.

Then the geodesic flow $\varphi^t : X \circlearrowleft$, $X = SM$ is hyperbolic on the entire X.

Recall that a map/flow is called <u>Anosov</u> if it is hyperbolic on the entire manifold.

<u>Thm</u> Assume that X is a compact connected manifold and $\varphi: X \to X$ is an Anosov diffeomorphism with a C^∞ invariant measure μ (i.e. μ has a C^∞ density w.r.t. Lebesgue).

Then φ is <u>mixing</u> w.r.t. μ.

In fact, it is <u>exponentially mixing</u>:

$\forall f, g \in C^\infty(X)$ ← need more than just L^2... Hölder is enough,

$$\int_X f(g \circ \varphi^n)\,d\mu - \left(\int_X f\,d\mu\right)\left(\int_X g\,d\mu\right) = O(e^{-\theta n})$$

as $n \to \infty$ for some fixed $\theta > 0$.

Not every <u>Anosov flow</u> is mixing (see Pset 2).

But <u>most of them are</u> (Anosov 1960s)

And we have

<u>**Thm**</u> [Dolgopyat '98, <u>Liverani '04</u>, … Tsujii '12, Nonnenmacher–Zworski '15]

Assume that X is a compact connected manifold and $\varphi^t : X \to X$ is a contact Anosov flow

(e.g. geodesic flow on $X = SM$
(M, g) negatively curved).

Then φ^t is <u>exponentially mixing</u>: $\forall f, g \in C^\infty(X)$ we have

$$\int_X f(g \circ \varphi^t)\, d\mu - \left(\int_X f\, d\mu\right)\left(\int_X g\, d\mu\right) \underset{t\to\infty}{=} O(e^{-\theta t})$$

where μ is the Liouville measure, $\theta > 0$.