§13. Patterson–Sullivan measures

We keep working with a convex co-compact hyperbolic surface $M = \Gamma \backslash \mathbb{H}^2$ where $\Gamma \subset PSL(2,\mathbb{R})$ is a <u>Schottky group</u>.

<u>Recall</u>: $\Lambda_\Gamma \subset \mathbb{R}$ the limit set of Γ, which is a compact set.

The group Γ acts on Λ_Γ.

Here we show (skipping some parts of the proof)

<u>Thm</u> There exist $\delta \in [0,1)$ and a probability measure μ on Λ_Γ such that $\forall f \in C^0(\Lambda_\Gamma)$, $\forall \gamma \in \Gamma$

$$\boxed{\int_{\Lambda_\Gamma} f \, d\mu = \int_{\Lambda_\Gamma} f(\gamma(x)) (\gamma'(x))^\delta \, d\mu(x)}$$

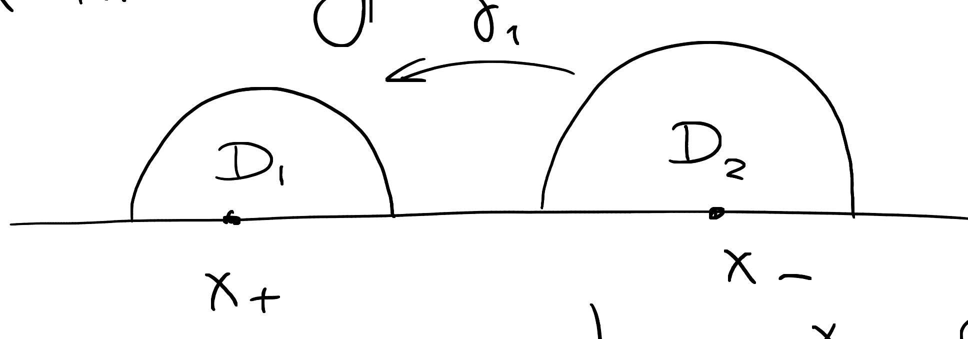
Here $\gamma'(x)$ is the derivative of $\gamma : \mathbb{R} \circlearrowleft$ at $x \in \mathbb{R}$.

Moreover, if M is not a hyperbolic cylinder then $0<\delta<1$ and μ with the above property is unique

---

μ is called (the) Patterson–Sullivan measure

Example: $M=$ hyperbolic cylinder,
$\Gamma = \{\gamma_1^j \mid j\in\mathbb{Z}\}$, $\gamma_1 \in PSL(2,\mathbb{R})$
a fixed hyperbolic element:



$\Lambda_\Gamma = \{x_+, x_-\}$ where $x_\pm$ are the fixed points of γ_1.
We have $\boxed{\delta=0}$ and μ can be any prob. measure on Λ_Γ:

$$\mu = \alpha\delta_{x_+} + (1-\alpha)\delta_{x_-}, \quad \alpha\in[0,1].$$

Indeed, e.g. for $\mu = \delta_{x_+}$
the equation

$$\int_{\Lambda_\Gamma} f\,d\mu = \int_{\Lambda_\Gamma} f(\gamma(x))\,\gamma'(x)^\delta\,d\mu(x)$$

becomes ($\gamma = \gamma_1^j$)

$$f(x_+) = f(\gamma_1^j(x_+))$$

which is true since $\gamma_1^j(x_+) = x_+ \quad \forall j$.

---

§13.1. Transfer operators

To construct μ, we reformulate the equivariance property in terms of transfer operators.

Recall that a Schottky group Γ is constructed using $2m$ disks $D_1, \dots, D_{2m}$ and maps $(\gamma_a)_{a\in\mathcal{A}}$, $\mathcal{A} = \{1, \dots, 2m\}$, with $\gamma_a(\dot{\mathbb{C}} \setminus D_{\bar{a}}) = D_a$, $\gamma_{\bar{a}} = \gamma_a^{-1}$,
$\bar{a} = a \pm m$.

Denote $C^0(\Lambda_\Gamma)$ = continuous functions on Λ_Γ.

For $s \in \mathbb{R}$, define the transfer operator

$\mathcal{L}_s : C^0(\Lambda_\Gamma) \to C^0(\Lambda_\Gamma)$ by

$$\mathcal{L}_s f(x) = \sum_{\substack{a \in \mathcal{A} \\ a \neq \bar{b}}} \left(\gamma_a'(x)\right)^s f(\gamma_a(x))$$

for all $f \in C^0(\Lambda_\Gamma)$, $x \in \Lambda_\Gamma \cap D_b$, $b \in \mathcal{A}$.

Note: Since $a \neq \bar{b}$ and $x \in D_b$, we have $x \notin D_{\bar{a}}$, so $\gamma_a(x) \in D_a$.

Example 1: hyperbolic cylinder

$\Lambda_\Gamma = \{x_+, x_-\}$, $x_+ \in D_1$, $x_- \in D_2$

$$\mathcal{L}_s f(x_+) = (\gamma_1'(x_+))^s f(x_+)$$

$$\mathcal{L}_s f(x_-) = (\gamma_1'(x_-))^s f(x_-)$$

Lemma Let μ be a probability measure on Λ_Γ and $s \in \mathbb{R}$. Then

μ is Γ-equivariant in the following sense

$$\int_{\Lambda_\Gamma} f\, d\mu = \int_{\Lambda_\Gamma} (f \circ \gamma)(\gamma')^s\, d\mu \quad \forall f \in C^0(\Lambda_\Gamma), \gamma \in \Gamma$$

$$\Updownarrow$$

μ satisfies $\mathcal{L}_s^* \mu = \mu$ in the following sense:

$$\int_{\Lambda_\Gamma} (\mathcal{L}_s f)\, d\mu = \int_{\Lambda_\Gamma} f\, d\mu \quad \forall f \in C^0(\Lambda_\Gamma).$$

Proof $\Downarrow$: Let $f \in C^0(\Lambda_\Gamma)$.

Let us compute (using that $\Lambda_\Gamma = \bigsqcup_{b \in \mathcal{A}} (\Lambda_\Gamma \cap D_b)$)

$$\int_{\Lambda_\Gamma} (\mathcal{L}_s f)\, d\mu = \sum_{b \in \mathcal{A}} \int_{\Lambda_\Gamma \cap D_b} (\mathcal{L}_s f)\, d\mu = \text{(by the definition of } \mathcal{L}_s\text{)}$$

$$= \sum_{b \in \mathcal{A}} \sum_{\substack{a \in \mathcal{A} \\ a \neq \bar{b}}} \int_{\Lambda_\Gamma \cap D_b} (\gamma_a'(x))^s f(\gamma_a(x))\, d\mu(x) = \text{(by } \Gamma\text{-equivariance of } \mu \text{ applied to } f \cdot 1_{D_{ab}},\ D_{ab} = \gamma_a(D_b)\text{)}$$

$$= \sum_{b \in \mathcal{A}} \sum_{\substack{a \in \mathcal{A} \\ a \neq \bar{b}}} \int_{\Lambda_\Gamma \cap D_{ab}} f\, d\mu = \ldots$$

$$\dots = \int_{\Lambda_\Gamma} f\,d\mu \quad \text{since } \Lambda_\Gamma = \bigsqcup_{\substack{a,b\in\mathcal{A}\\ a\neq\bar{b}}} D_{ab}.$$

18.118
13-6

$\Uparrow$: For any $f \in C^0(\Lambda_\Gamma)$, $a,b\in\mathcal{A}$, $a\neq\bar{b}$

applying the equality $\mathcal{L}_s^* \mu = \mu$ to the function $1_{D_{ab}} \cdot f$, we get

$$\int_{D_{ab}} f\,d\mu = \int_{\Lambda_\Gamma} (1_{D_{ab}} f)\,d\mu = \int_{\Lambda_\Gamma} \mathcal{L}_s (1_{D_{ab}} f)\,d\mu$$

$$= \int_{D_b} (f\circ\gamma_a)(\gamma_a')^s\,d\mu$$

Since $\mathcal{L}_s (1_{D_{ab}} f)(x) = \begin{cases} f(\gamma_a(x))\,\gamma_a'(x)^s, & x\in D_b \\ 0, & x\notin D_b. \end{cases}$

Now, we need to show $\forall \gamma\in\Gamma$, $f\in C^0(\Lambda_\Gamma)$ the identity

$$(*) \quad \int_{\Lambda_\Gamma} f\,d\mu = \int_{\Lambda_\Gamma} (f\circ\gamma_a)(\gamma_a')^s\,d\mu.$$

18.118
13-7

- If $(*)$ holds for some $\gamma, \tilde\gamma \in \Gamma$ & all $f \in C^0(\Lambda_\Gamma)$ then it also holds for $\gamma\tilde\gamma$ and all f:

$$\int_{\Lambda_\Gamma} f\,d\mu \overset{(*)\text{ for }\gamma}{=} \int_{\Lambda_\Gamma} (f\circ\gamma)(\gamma')^s\,d\mu \overset{(*)\text{ for }\tilde\gamma}{=}$$

$$= \int_{\Lambda_\Gamma} (f\circ\gamma\circ\tilde\gamma)(\gamma'\circ\tilde\gamma)^s(\tilde\gamma')^s\,d\mu$$

$$= \int_{\Lambda_\Gamma} (f\circ\gamma\tilde\gamma)((\gamma\tilde\gamma)')^s\,d\mu.$$

Since Γ is generated by $(\gamma_a)_{a\in\mathcal{A}}$ it suffices to check $(*)$ for all γ_a.

- If $(*)$ holds for γ^{-1} and $(f\circ\gamma)(\gamma')^s$ then it also holds for γ and f: indeed, $(*)$ for γ^{-1} and $(f\circ\gamma)(\gamma')^s$ gives

$$\int_{\Lambda_\Gamma}(f\circ\gamma)(\gamma')^s\,d\mu = \int_{\Lambda_\Gamma}(f\circ\gamma\circ\gamma^{-1})(\gamma'\circ\gamma^{-1})^s((\gamma^{-1})')^s\,d\mu$$

$= \int_{\Lambda_\Gamma} f\,d\mu$, so $(*)$ holds for γ and f.

18.118
13-8

- If $a \in \mathcal{A}$ and $\operatorname{supp} f \subset D_a$ then (*) holds for γ_a and f:

$$\int_{\Lambda_\Gamma} f \, d\mu = \int_{D_a} f \, d\mu = \sum_{b \neq \bar{a}} \int_{D_{ab}} f \, d\mu =$$

$$= \int_{D_b} (f \circ \gamma_a)(\gamma_a')^s \, d\mu.$$

- If $a \in \mathcal{A}$ and $\operatorname{supp} f \cap D_a = \emptyset$ then (*) holds for γ_a and f.

Indeed, it suffices to show that (*) holds for $\gamma_{\bar{a}} = \gamma_a^{-1}$ and $\tilde{f} := (f \circ \gamma_a)(\gamma_a')^s$.

But $\operatorname{supp} \tilde{f} = \gamma_a^{-1}(\operatorname{supp} f) \subset$
$\subset \gamma_{\bar{a}}(\mathbb{C} \setminus D_a) = D_{\bar{a}}$, so this reduces to the previous case (with a replaced by $\bar{a}$)

- The previous 2 cases show that (*) holds for all γ_a, $a \in \mathcal{A}$, and all f.

$\square$

§ 13.2. Construction of the measure

Here we present a particular version.
For the general case see e.g.
Bowen, "Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms", Section 1.7

What we describe below is part of the Ruelle-Perron-Frobenius Theorem, see Bowen.

Lemma Let $s \in \mathbb{R}$. Then $\exists$ a probability measure μ_s on Λ_Γ and a number $\lambda_s > 0$ such that

$$\int_{\Lambda_\Gamma} (\mathcal{L}_s f)\, d\mu_s = \lambda_s \cdot \int_{\Lambda_\Gamma} f\, d\mu_s \quad \forall f \in C^0(\Lambda_\Gamma).$$

Proof (sketch)

18.118
13-10

Let $\mathcal{M}$ be the space of probability measures on Λ_Γ, with weak convergence.

By the Riesz Representation Thm and Compactness Thm we can think of $\mathcal{M}$ as a compact convex subset of the dual space to $C^0(\Lambda_\Gamma)$.

Define the following operator $T_s: \mathcal{M} \mathfrak{S}$:

for any $\mu \in \mathcal{M}$ and $f \in C^0(\Lambda_\Gamma)$ we have
$$\int_{\Lambda_\Gamma} f \, d(T_s\mu) = \frac{\int_{\Lambda_\Gamma} L_s f \, d\mu}{\int_{\Lambda_\Gamma} L_s 1 \, d\mu}.$$

Here $T_s\mu$ is a prob. measure by the Riesz Representation Thm, as the functional
$$f \mapsto \frac{\int_{\Lambda_\Gamma} L_s f \, d\mu}{\int_{\Lambda_\Gamma} L_s 1 \, d\mu}$$

is linear in f,

nonnegative when $f \geq 0$ (since then $\mathcal{L}_s f \geq 0$ too)

and sends $f \equiv 1$ to 1

(here $\int_{\Lambda_\Gamma} \mathcal{L}_s 1 \, d\mu > 0$ since $\mathcal{L}_s 1 > 0$ on Λ_Γ)

Moreover, $T_s : \mathcal{M} \circlearrowleft$ is <u>continuous</u> w.r.t. weak topology on $\mathcal{M}$.

We now use the <u>Schauder-Tychonoff Thm</u> (see Dunford-Schwartz, Linear Operators I, p. 456) which itself is a generalization of the Brouwer fixed point theorem, to see that T_s has a fixed point:

$\exists \mu_s \in \mathcal{M} : \quad T_s(\mu_s) = \mu_s$.

Now μ_s has the needed property, with $\lambda_s := \int_{\Lambda_\Gamma} \mathcal{L}_s(1) \, d\mu_s$. $\square$

18.118
13-12

The map $s \in \mathbb{R} \mapsto \lambda_s \in (0,\infty)$ is continuous (we won't prove this here...).

Note that

$$\boxed{s=0 \Rightarrow \lambda_0 \geq 1.}$$

Indeed, recalling that

$$L_s f(x) = \sum_{\substack{a\in\mathcal{A} \\ a\neq \bar{b}}} (\gamma_a'(x))^s f(\gamma_a(x)), \quad x\in\Lambda_r\cap D_b$$

we have for $s=0$, $L_s(1)\geq 1$.

In fact, if M is not a hyperbolic cylinder then the above sum has $2m-1\geq 3$ elements for each x, so $\lambda_0>1$

We will soon show that

$$\boxed{S=1 \Rightarrow \lambda_1 \leq 1}$$

Then by Intermediate Value Theorem $\exists\, \delta : \lambda_\delta = 1$, and μ_δ will be the Patterson-Sullivan measure.

A few basic properties first:

- Exponential contraction: $\exists C, \theta > 0$: $\forall \vec{a} \in \mathcal{W}^n$, $\operatorname{diam}(D_{\vec{a}}) \le C e^{-\theta n}$

Will skip the proof; see e.g. Borthwick, Prop. 15.5

- Bounded distortion: if $\vec{a} = a_1 \dots a_n \in \mathcal{W}^n$, $b \ne \bar{a}_n$, and $x, y \in I_b := D_b \cap \mathbb{R}$

$$\gamma'_{\vec{a}}(x) \le C \gamma'_{\vec{a}}(y)$$

where C is independent of n.

Proof Use the Chain Rule similarly to Pset 4, Problem 2:

$$\frac{\gamma_{\vec{a}}(x)}{\gamma_{\vec{a}}(y)} = \prod_{j=1}^{n} \frac{\gamma'_{a_j}(\gamma_{a_{j+1} \cdots a_n}(x))}{\gamma'_{a_j}(\gamma_{a_{j+1} \cdots a_n}(y))}$$

Now, $\forall j$ we have

$\gamma_{a_{j+1}\cdots a_n}(x), \gamma_{a_{j+1}\cdots a_n}(y) \in D_{a_{j+1}\cdots a_n b}$.

So by the exponential contraction property,

$$|\gamma_{a_{j+1}\cdots a_n}(x) - \gamma_{a_{j+1}\cdots a_n}(y)| \leq Ce^{-\theta(n-j)}.$$

Thus $$\frac{\gamma'_{a_j}(\gamma_{a_{j+1}\cdots a_n}(x))}{\gamma'_{a_j}(\gamma_{a_{j+1}\cdots a_n}(y))} \leq 1 + Ce^{-\theta(n-j)}$$

and the product $\prod_{j=1}^{n}(1+Ce^{-\theta(n-j)})$

is bounded above uniformly in n.

- From bounded distortion we get:

if $I_{\vec{a}} = D_{\vec{a}} \cap \mathbb{R}$ and $|I_{\vec{a}}|$

denotes the length of $I_{\vec{a}}$, then ($\vec{a} = a_1 \dots a_n$)

$\forall x \in I_{a_n}$, $\vec{a}' := a_1 \dots a_{n-1}$,

$$C^{-1}|I_{\vec{a}}| \leq \gamma'_{\vec{a}'}(x) \leq C|I_{\vec{a}}|.$$

Indeed, $|I_{\vec{a}}| = |\gamma_{\vec{a}'}(I_{a_n})| = \int_{I_{a_n}} \gamma'_{\vec{a}'}(x)\,dx$.

- Coming back to the measure μ_s such that $\mathcal{L}_s^* \mu_s = \lambda_s \mu_s$: from the definition of $\mathcal{L}_s$ we have $\forall f \in C^0(I_{ab})$, $a \neq \bar{b}$,

$$\int_{I_{ab}} f \, d\mu_s = \lambda_s^{-1} \int_{I_b} (f \circ \gamma_a)(\gamma_a')^s \, d\mu_s .$$

Indeed, $\mathcal{L}_s(f \mathbb{1}_{I_{ab}}) =$

$$= (f \circ \gamma_a)(\gamma_a')^s \mathbb{1}_{I_b} .$$

- Iterating the above, we see that $\forall \vec{a} = a_1 \dots a_n \in W^n$, $\vec{a}' = a_1 \dots a_{n-1}$, $\forall f \in C^0(I_{\vec{a}})$, we have

$$\int_{I_{\vec{a}}} f \, d\mu_s = \lambda_s^{1-n} \int_{I_{a_n}} (f \circ \gamma_{\vec{a}'})(\gamma_{\vec{a}'}')^s \, d\mu_s .$$

Taking $f \equiv 1$ in this identity, we see that $\forall \vec{a} = a_1 \dots a_n \in W^n$

$\vec{a}' := a_1 \dots a_{n-1}$,

$$\mu_s(I_{\vec{a}}) = \lambda_s^{1-n} \int_{I_{a_n}} \left(\gamma_{\vec{a}'}'(x)\right)^s d\mu_s .$$

We previously showed that

$$\gamma_{\vec{a}}'(x) \sim |I_{\vec{a}}| \text{ for } x \in I_{a_n}.$$

Thus $\exists C = C(s) \forall \vec{a}$ (if M is not a hyperbolic cylinder)

$$\boxed{C^{-1} \frac{|I_{\vec{a}}|^s}{\lambda_s^n} \leq \mu_s(I_{\vec{a}}) \leq C \frac{|I_{\vec{a}}|^s}{\lambda_s^n}}$$

(Here we also used that $\mu_s(I_a) > 0$ $\forall a \in \mathcal{A}$.

This is true since M is not a hyperbolic cylinder: if $\mu_s(I_a) = 0$ for some a, then $\mu_s(I_{ab}) = 0$ $\forall b \neq \bar{a}$ (as $I_{ab} \subset I_a$) but then by the above formula $\mu(I_b) = 0$ $\forall b \neq \bar{a}$. from this we can get $\mu(I_b) = 0$ $\forall b$, a contradiction as $\mu(\bigsqcup_b I_b) = 1$.)

· Now let's take $s=1$.

We have: $\exists C \ \forall \vec{a} \in W^n$

$$\mu_s(I_{\vec{a}}) \le C \frac{|I_{\vec{a}}|}{\lambda_1^n}.$$

Since $\Lambda_\Gamma = \bigsqcup_{\vec{a} \in W^n} (\Lambda_\Gamma \cap I_{\vec{a}})$, we get

$$1 = \sum_{\vec{a} \in W^n} \mu_s(I_{\vec{a}}) \le \frac{C}{\lambda_1^n} \sum_{\vec{a} \in W^n} |I_{\vec{a}}|.$$

But $\sum_{\vec{a} \in W^n} |I_{\vec{a}}| \le C$

since $I_{\vec{a}}$ are nonintersecting & lie inside $\bigsqcup_{a \in A} I_a$, so

$$1 \le \frac{C}{\lambda_1^n} \Rightarrow \lambda_1^n \le C \underset{\text{taking } n \to \infty}{\Rightarrow} \boxed{\lambda_1 \le 1}$$

With a bit more work, can show that

$\sum_{\vec{a} \in W_n} |I_{\vec{a}}| \longrightarrow 0$ as $n \to \infty$ and thus

$$\boxed{\lambda_1 < 1}$$

- We can now construct the Patterson-Sullivan measure:

- $s \mapsto \lambda_s$ is continuous,
- $\lambda_0 \geq 1$,
- $\lambda_1 < 1$.
- By the Intermediate Value Thm there exists $\boxed{\delta \in [0,1)}$ such that $\boxed{\lambda_\delta = 1}$.

Put $\mu = \mu_\delta$, then $\mathcal{L}_\delta^* \mu_\delta = \mu_\delta$, which shows that μ_δ satisfies the Γ-equivariance property.

Thus μ_δ is a Patterson-Sullivan measure
(won't prove uniqueness...)

If M not a hyp. cylinder, then $\lambda_0 > 1$ and thus $\boxed{0 < \delta < 1}$

§ 13.3. Further properties

Now we have constructed the Patterson-Sullivan measure μ and defined $\delta \in [0,1)$ (with $\delta = 0 \iff M$ is a hyperbolic cylinder).

From the discussion in § 13.2 we see that $\exists C \; \forall \vec{a} \in \mathcal{W}^n$,

$$C^{-1} |I_{\vec{a}}|^\delta \leq \mu(I_{\vec{a}}) \leq C |I_{\vec{a}}|^\delta .$$

This is a version of Ahlfors-David regularity.

From here one can deduce (see Borthwick Thm 14.14)

<u>Thm</u> (Patterson-Sullivan)

The Hausdorff dimension of Λ_Γ is equal to δ.

<u>Corollary</u>: the Hausdorff dimension of the set K of trapped geodesics on $M = \Gamma \backslash \mathbb{H}^2$ is equal to $2\delta + 1$

(as locally, $K \simeq \Lambda_\Gamma \times \Lambda_\Gamma \times \mathbb{R}$)

We can use the Patterson-Sullivan measure μ to define a measure $\tilde{\mu}$ on the trapped set

$$K \subset SM.$$

Recall from §12.4 that, denoting by $\tilde{\pi}: S\mathbb{H}^2 \to SM$ the projection map, we have

$$\tilde{\pi}^{-1}(K) \simeq (\Lambda_\Gamma \times \Lambda_\Gamma)_\Delta \times \mathbb{R}$$

by the map $S\mathbb{H}^2 \to (\nu_-, \nu_+, s)$

and in this identification, the geodesic flow φ^t is just $(\nu_-, \nu_+, s) \mapsto (\nu_-, \nu_+, t+s)$.

Now, define the measure $\tilde{\mu}$ on $\tilde{\pi}^{-1}(K)$ by

$$d\tilde{\mu} = \frac{d\mu(\nu_-)\, d\mu(\nu_+)\, ds}{|\nu_- - \nu_+|^{2\delta}}.$$

Using Γ-equivariance of μ, one can show (see e.g. Borthwick, §14.2)

that $\tilde{\mu}$ is invariant under the action of Γ on $S\mathbb{H}^2$.

Thus it descends to a measure $\tilde{\mu}$ on $K \subset SM$, which we multiply by a constant to make into a probability measure.

The measure $\tilde{\mu}$ on K is invariant under φ^t and, if M is not a hyperbolic cylinder, one can show (see Borthwick for some of these):

- φ^t is mixing w.r.t. $\tilde{\mu}$
- $h_{top}(\varphi^t|_K) = h_{\tilde{\mu}}(\varphi^t) = \delta$, in particular $\tilde{\mu}$ is a measure of maximal entropy