Torsion subgroups of abelian surfaces over $\mathbb{Q}$ (products)

There are 96 distinct products of $E/\mathbb{Q}$ torsion subgroups; of which 21 are maximal:

$$C_{63}, \quad C_{70}, \quad C_{84}, \quad C_{90},$$

$$C_2 \times C_{42}, \quad C_2 \times C_{56}, \quad C_2 \times C_{60}, \quad C_2 \times C_{72},$$

$$C_3 \times C_{36}, \quad C_6 \times C_{18},$$

$$C_7^2, \quad C_9^2, \quad C_{10}^2, \quad C_{12}^2$$

$$C_2 \times C_2 \times C_{30}, \quad C_2 \times C_2 \times C_{40},$$

$$C_2 \times C_4 \times C_{24}, \quad C_2 \times C_6 \times C_{12}$$

$$C_2 \times C_2 \times C_2 \times C_{24},$$

$$(C_2 \times C_6)^2, \quad (C_2 \times C_8)^2$$

Every subgroup of these groups occurs (not necessarily in a Jacobian; see Table 1 of Howe-Leprévost-Poonen for examples known to arise in Jacobians).

Torsion subgroups of abelian surfaces over $\mathbb{Q}$ (split)

In addition to torsion subgroups that arise for products of elliptic curves, there are torsion subgroups that arise for abelian surfaces that are geometrically isogenous to a product of elliptic curves but not isomorphic to a product of elliptic curves over $\mathbb{Q}$

The following groups arise for abelian surfaces (in fact, Jacobians) that are $\mathbb{Q}$-isogenous to a product of elliptic curves but do not arise in products of elliptic curves:

$$C_{27}, \quad C_{48}, \quad C_2 \times C_{48}, \quad C_3 \times C_{24}, \quad C_4 \times C_{16}, \quad C_2 \times C_2 \times C_{16}$$

The following groups arise for $\mathbb{Q}$-simple, geometrically split abelian surfaces (in fact, Jacobians) defined over $\mathbb{Q}$, but are not known to arise for $\mathbb{Q}$-split abelian surfaces:

$$C_{19}, \quad C_{25}$$

Known torsion subgroups of abelian surfaces over $\mathbb{Q}$ (geometrically simple)

There are (at least) 63 torsion subgroups known to arise for geometrically simple abelian surfaces (in fact, Jacobians) defined over $\mathbb{Q}$, including:

$$C_n \text{ for } 1 \le n \le 30, 32, 33, 34, 36, 39, 40$$

$$C_2 \times C_{2n} \text{ for } 1 \le n \le 9, 11, 13, 14$$

$$C_2 \times C_2 \times C_{2n} \text{ for } 1 \le n \le 7$$

$$C_2 \times C_2 \times C_2 \times C_{2n} \text{ for } 1 \le n \le 3, 5$$

$$C_3 \times C_{3n} \text{ for } 1 \le n \le 3,$$

$$C_4 \times C_4$$

Generic abelian surfaces (no extra endomorphisms over $\overline{\mathbb{Q}}$) that realize these groups are known in all but possibly two cases: $C_2 \times C_{22}$ and $C_2 \times C_2 \times C_{14}$.

WARNING: This list is provisional and is very likely to be missing some known cases.