18.315 Problem Set 5 (due Tuesday, December 12, 2006.)

Turn in at most 5 problems.

1. A *domino tableau* is a ribbon tableau of weight $(2, 2, \ldots, 2)$. Let $DT(\lambda)$ be the number of domino tableaux of shape λ. Find a closed expression for the sum $\sum_\lambda DT(\lambda)^2$ over partitions λ such that $|\lambda| = 2n$.

2. For partitions λ, μ, γ with k parts, prove that the Kostka number equals the Littlewood-Richardson coefficient $K_{\lambda\mu} = c^{\mu+\gamma}_{\lambda,\gamma}$, if γ satisfies the condition $\min|\gamma_i - \gamma_{i+1}| > \mu_1$. (For example, the equality $K_{\lambda\mu} = c^{\mu+\gamma}_{\lambda,\gamma}$ holds if $\gamma = (kN, (k-1)N, \ldots, N)$ for sufficiently large N.)

3. Construct a bijection between two variants BZ_1 and BZ_2 of Berenstein-Zelevinsky triangles. (BZ_1 involves the hexagon condition and BZ_2 has the tail-sum condition.)

4. Construct a bijection between the set of Littlewood-Richardson tableaux $LR(\lambda/\mu, \nu)$ and the set of Knutson-Tao honeycombs with boundary rays given by λ, μ, and ν (as described in class).

5. Prove Knutson-Tao's puzzle version of the LR-rule. (It is enough to show that the puzzle LR-rule is equivalent to another version: LR-tableaux, BZ-triangles, or KT-honeycombs.)

6. Let $(\lambda/\mu)^\vee$ be the skew shape λ/μ rotated by 180°. Construct a bijection between the sets of Littlewood-Richardson tableaux $LR(\lambda/\mu, \nu)$ and $LR((\lambda/\mu)^\vee, \nu)$.

7. Let V_λ be the irreducible representations of S_n labelled by partitions λ as in the Okounkov-Vershik construction. (That is the eigenvalues of the Jucys-Murphy elements in the representation V_λ are the contents of the shape λ.) Also let $\tilde{V}_\lambda$ be the irreducible representation of S_n whose character χ_λ corresponds to the Schur function s_λ under the Frobenius characteristic map ch. Prove that $V_\lambda = \tilde{V}_\lambda$. In other words, show that Okounkov-Vershik's and Frobenius' approaches lead to the same labelling of the irreducible representations by partitions.

8. (a) Prove that $\sum_\lambda z_\lambda^{-1}\, p_\lambda(x)\, p_\lambda(y) = \prod_{i,j} \frac{1}{1-x_i y_j}$ and deduce that $\langle p_\lambda, p_\mu\rangle = z_\lambda\, \delta_{\lambda\mu}$. (b) Prove that $\sum_{\lambda:\,|\lambda|=n} z_\lambda^{-1}\, p_\lambda = h_n$.

9. Calculate all values of the character $\chi_{(n-1,1)}$ of the irreducible representation $V_{(n-1,1)}$ of S_n.

10. For a partition λ, give a closed formula for the character value $\chi_\lambda((2, 1, \ldots, 1))$.