18.315 Problem Set 4 (due Thursday, November 16.)

1. Express the power symmetric function $p_k = x_1^k + x_2^k + x_3^k + \cdots$ as a linear combination of Schur functions s_λ.

2. (a) Show that $s_\lambda \cdot p_2 = \sum_\mu s_\mu - \sum_\nu s_\nu$, where the first sum is over partitions μ obtained from λ by adding a horizontal domino and the second sum is over ν obtained from λ by adding a vertical domino.

(b) Let us encode a Young diagram λ by the 01-sequence, infinite in both directions, where 1's correspond to vertical steps and 0's correspond to horizontal steps as we go North-East along the border of λ. For example, $\lambda = (3, 1)$ is encoded as $\cdots 111101001000 \cdots$. Show that adding a horizontal or a vertical domino to λ corresponds to switching 1 and 0 two steps apart in the 01-sequence.

(c) Characterize Young diagrams λ that can be subdivided into horizontal and vertical dominos.

(d) Prove that the parity of the number of vertical dominos is the same for any two domino subdivisions of the same Young diagram λ.

(e) Expand the product $(p_2)^n$ as a linear combination of Schur functions s_λ. (In other words, for any λ, calculate the coefficient of s_λ in this expansion.) For example, $(p_2)^2 = s_{(4)} - s_{(3,1)} + 2s_{(2,2)} - s_{(2,1)} + s_{1^4}$.

(f)* Generalize parts (a)–(e) to p_k and expand the product $(p_k)^n$ as a linear combination of Schur functions.

3. Prove the following hooklength-content formula:

$$s_\lambda(1, q, q^2, \ldots, q^{n-1}) = q^{m(\lambda)} \prod_{x \in \lambda} \frac{1 - q^{n+c(x)}}{1 - q^{h(x)}} = q^{m(\lambda)} \prod_{x \in \lambda} \frac{[n + c(x)]_q}{[h(x)]_q}$$

where $c(x) = j - i$ is the *content* and $h(x)$ is the *hooklength* of a box $x = (i, j)$ in λ, and $m(\lambda) = \sum (i - 1)\lambda_i$.

4. Let $A(n, k)$ be the number of reverse plane partitions of the triangular shape $\rho = (n - 1, n - 2, \ldots, 1)$ with entries $\leq k$.

(a) Express the number $A(n, k)$ as the determinant of a certain $k \times k$-matrix (with entries given by explicit expressions).

(b) Find the limit $\lim_{k\to\infty} \frac{A(n,k)}{k^N}$, where $N = |\rho| = \binom{n}{2}$.

5. We have constructed the three bijections f_{RSK}, f_{HG}, and f_V between the set of $n \times n$-matrices with nonnegative integer entries and the set of reverse plane partitions of the square shape $n \times n$, where f_{RSK} is the RSK correspondence written via Gelfand-Tsetlin patterns (see Problem 8 from Problem Set 2); f_{HG} is the Hillman-Grassl correspondence; and f_V is the Viennot correspondence. Is there any relation between these bijections?

6. For a partition λ with $|\lambda| = n$. Let $P_{\lambda,N}$ be the polytope of points $(x_{ij})_{(i,j)\in\lambda} \subset \mathbb{R}^n$ given by the conditions $x_{ij} \geq 0$; $x_{ij} \leq x_{i'j'}$ for $i \leq i'$ and $j \leq j'$; $\sum x_{ij} \leq N$.

(a) Prove that $\mathrm{Vol}(P_{\lambda,1}) = f^\lambda\,\mathrm{Vol}(\Delta)$, where f^λ is the number of standard Young tableaux of shape λ and Δ is the simplex $\{(x_1,\dots,x_n) \in \mathbb{R}^n \mid 0 \leq x_1 \leq \cdots \leq x_n;\ \sum x_i \leq 1\}$.

(b) We proved (using the Hillman-Grassl correspondence) that the sum $\sum_T q^{\sum T_{ij}}$ over reverse plane partitions T of shape λ equals $\prod_{x\in\lambda}(1-q^{h(x)})^{-1}$. Deduce that the number $\#(P_{\lambda,N}\cap\mathbb{Z}^n)$ of integer lattice points in $P_{\lambda,N}$ equals the N-th coefficient in the Taylor expansion of $(1-q)^{-1}\prod_{x\in\lambda}(1-q^{h(x)})^{-1}$.

(c) Use (a), (b), and the fact that $\mathrm{Vol}(P_{\lambda,1}) = \lim \frac{\#(P_{\lambda,N}\cap\mathbb{Z}^n)}{N^n}$ to deduce the hook-length formula $f^\lambda = \frac{n!}{\prod_{x\in\lambda} h(x)}$.

7. Let $u_1,\dots,u_n$ be some noncommutative variables. For a tableau T with entries $a_1, a_2, \dots, a_N \in \{1,\dots,n\}$ listed by columns bottom-up, left-to-right, let $u^T := u_{a_1}u_{a_2}\dots u_{a_N}$. For example, for $T = \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&3\\\cline{1-2}\end{array}$, $u^T = u_2u_1u_3u_1u_3$. The *noncommutative Schur polynomial* $S_\lambda(u_1,\dots,u_n)$ is defined as the sum $\sum u^T$ over semi-standard tableaux of shape λ.

Let Y_i, $i\in\mathbb{Z}$ be the (noncommutative) operators acting on the space linear combinations of Young diagrams by $Y_i(\lambda) = \mu$ if μ is obtained from λ by adding a box to the ith diagonal; otherwise (if it is impossible to add a box to the i-th diagonal of λ), set $Y_i(\lambda) = 0$. For example $Y_1(\begin{array}{|c|c|c|}\hline \ &\ &\ \\\hline \ &\ \\\cline{1-2}\end{array}) = \begin{array}{|c|c|c|}\hline \ &\ &\ \\\hline \ &\ &\ \\\hline\end{array}$ and $Y_0(\begin{array}{|c|c|c|}\hline \ &\ &\ \\\hline \ &\ \\\cline{1-2}\end{array}) = 0$.

(a) Show that $S_\lambda(Y_{-k+1}, Y_{-k+1},\dots,Y_{l-1})\cdot\emptyset = \lambda$ if λ fits inside the $k\times l$-rectangle.

(b) Prove that the operators Y_i satisfy the relations $Y_i^2 = Y_i\,Y_{i+1}\,Y_i = Y_{i+1}\,Y_i\,Y_{i+1} = 0$ and $Y_i\,Y_j = Y_j\,Y_i$ when $|i-j|\geq 2$.

(c) Let $H_k = \sum_{i_1\leq i_2\leq\cdots\leq i_k} Y_{i_1}\cdots Y_{i_k}$ and $E_k = \sum_{j_1>j_2>\cdots>j_k} Y_{j_1}\cdots Y_{j_k}$. Also set $H_0 = E_0 = 1$. Prove (using only relations from (b)) that all operators $E_1, E_2, \dots, H_1, H_2, \dots$ commute with each other. Also prove that $E_k\,H_0 - E_{k-1}\,H_1 + E_{k-2}\,H_2 - \cdots + (-1)^k\,E_0\,H_k = 0$.

(d) Prove the following "noncommutative Cauchy identities":

$$\prod_{i=1}^{m}\prod_{j=-k+1}^{l-1}\frac{1}{1-x_iY_j} = \sum_\lambda s_\lambda(x_1,\dots,x_m)\,S_\lambda(Y_{-k+1},\dots,Y_{l-1})$$

$$\prod_{i=1}^{m}\prod_{j=l-1}^{-k+1}(1+x_iY_j) = \sum_\lambda s_{\lambda'}(x_1,\dots,x_m)\,S_\lambda(Y_{-k+1},\dots,Y_{l-1})$$

where the sums are over λ fitting inside the $k\times l$-rectangle. (Note the order of terms in the product.) Here $x_1, \dots, x_m$ are commutative variables (which commute with each other and with Y_j's) and $s_\lambda(x_1, \dots, x_m)$ are the usual Schur functions.

8. A *reduced decomposition* of a permutation $w \in S_n$ is a way to write w as a product of adjacent transpositions $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ of minimal possible length l (= the number of inversions in w). For example, the permutation $(3, 2, 1) \in S_3$ (written in one-line notation) has two reduced decompositions: $s_1 s_2 s_1$ and $s_2 s_1 s_2$.

(a) For $n > k \geq 1$, calculate the number of reduced decompositions of the permutation $(k+1, k+2, \dots, n, 1, 2, \dots, k) \in S_n$.

(b) Calculate the number of reduced decompositions of the maximal permutation $(n, n-1, \dots, 2, 1) \in S_n$.

(c) Say that two reduced decompositions are *commutation equivalent* if they can be obtained from each other by a sequence of switches of adjacent entries $s_i s_j \to s_j s_i$ for $|i - j| \geq 2$. (Equivalently, two reduced decomposition are commutation equivalent if the corresponding *wiring diagrams* are homotopy equivalent.) For example, $s_2 s_1 s_3$ is commutation equivalent to $s_2 s_3 s_1$.

For positive integers k, l, m such that $k + l + m = n$, calculate the number of classes of commutation equivalent reduced decompositions of the permutation $(k + l + 1, k + l + 2, \dots, n, k + 1, k + 2, \dots, k + l, 1, 2, \dots, k) \in S_n$.

9. (a) Let A be a pseudoline arrangement with n pseudolines such that every pair of pseudolines have exactly one intersection. Prove that A has at least $n - 2$ triangular regions.

(a)$'$ You can try to prove a slightly weaker version of this claim for a line arrangement.

(b) Construct a pseudoline arrangement which is not a line arrangement. (That is the pseudolines cannot be straightened.)

10. For a poset P, let C_k be the maximal number of elements in a union of k chains, and let A_k be the maximal number of elements in a union of k antichains. Prove that $\lambda = (C_1, C_2 - C_1, C_3 - C_2, \dots)$ and $\mu = (A_1, A_2 - A_1, A_3 - A_2, \dots)$ are partitions which are conjugate to each other: $\lambda' = \mu$.

In particular, this implies that if k is the maximal size an antichain (respectively, the maximal size of a chain) in P, than P can be subdivided into k chains (respectively, antichains).