18.315 PROBLEM SET 3 (due Thursday, October 26, 2006)

Turn in at most 6 problems.

1. Prove the following identity for q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{r=0}^{\min(k,n-k)} q^{r^2} \begin{bmatrix} k \\ r \end{bmatrix}_q \begin{bmatrix} n-k \\ r \end{bmatrix}_q$$

2. For any $n \geq k \geq 0$, calculate explicitly the value of the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ at $q = -1$.

3. (a) For a nonempty partition λ, prove that the skew Schur function $s_{\lambda/1}$ equals the sum of Schur functions s_μ over all partitions μ obtained from λ by removing a corner box.

(b) For a partition λ whose Young diagram has at least two rows and at least two columns, prove that $s_{\lambda/(2)} - s_{\lambda/(1^2)}$ equals $\sum s_\mu - \sum s_\nu$ over all partitions μ obtained from λ by removing a horizontal domino and all partitions ν obtained from λ by removing a vertical domino.

(c) Find all partitions λ such that $s_{\lambda/(2)} = s_{\lambda/(1^2)}$.

4. For a positive integer n and a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, let $\#SSYT(\lambda, n)$ be the total number of semi-standard Young tableaux of shape λ filled with entries $\leq n$. Calculate the generating function

$$F_n(q) = \sum_{\lambda=(\lambda_1,\dots,\lambda_n)} \#SSYT(\lambda, n)\, q^{|\lambda|},$$

where the sum is over partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ (with fixed n). For example, $F_1(q) = 1/(1-q)$ and $F_2(q) = 1/((1-q)^2(1-q^2))$.

5. Find a closed formula for the number $\#SSYT(\lambda, n)$ of semi-standard Young tableaux of shape λ filled with entries $\leq n$ (see the previous problem).

6. For a strict partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0)$, the *shifted Young diagram* of shape λ is the collection of boxes with coordinates $\{(i,j) \mid i = 1, \dots, l;\ j = i, i+1, \dots, i+\lambda_i\}$. Let $S_{k,n}$ be the number of shifted Young diagrams such that $\lambda_1 \leq n$ and $|\lambda| = \lambda_1 + \cdots + \lambda_l = k$. Prove that the sequence, $S_{0,n}, S_{1,n}, \dots, S_{N,n}$ (where $N = n(n+1)/2$) is unimodal, that is

$$S_{0,n} \leq S_{1,n} \leq \cdots \leq S_{\lfloor N/2 \rfloor,n} \geq \cdots \geq S_{N,n}.$$

7. Prove that $(x_1 + x_2 + x_3 + \cdots)^n = \sum s_\kappa$, where the sum of skew Schur functions is over all 2^{n-1} ribbons κ with n boxes.

8. For a subset $I \subseteq [n-1]$, let $\beta(I)$ be the number of permutations $w \in S_n$ with the set of descents I, that is permutations w such that $w_i > w_{i+1}$ if $i \in I$ and $w_j < w_{j+1}$ if $j \in [n-1] \setminus I$. Let $S(I) := \{j \in [n-2] \mid \#(\{j, j+1\} \cap I) = 1\}$. Prove that, for two subsets $I, J \subseteq [n-1]$, if $S(I) \supseteq S(J)$, then $\beta(I) \geq \beta(J)$.

9. Show that for 3 partititions λ, μ, ν such that $|\lambda| = |\mu| = |\nu|$, we have $K_{\lambda,\mu} \leq K_{\lambda,\nu}$ when $\mu \geq \nu$ in the *dominance order,* that is $\mu_1 + \cdots + \mu_i \geq \nu_1 + \cdots + \nu_i$, for $i = 1, 2, \ldots$.

10. We constructed in class, the operation $\tilde{s}_i$ acting on semi-standard Young tableaux T that swaps in the number of i's and $(i+1)$'s in T. (This operation is called the *Bender-Knuth involution.*) Let $q_i := (\tilde{s}_1 \tilde{s}_2 \cdots \tilde{s}_i)(\tilde{s}_1 \tilde{s}_2 \cdots \tilde{s}_{i-1})(\tilde{s}_1 \tilde{s}_2 \cdots \tilde{s}_{i-2}) \cdots (\tilde{s}_1)$. Let $s_i := q_i \, \tilde{s}_1 \, (q_i)^{-1}$. Show that the operators s_i satisfy the Coxeter relations: $(s_i)^2 = (s_i s_j)^2 = (s_i s_{i+1})^3 = 1$, where $j \neq i \pm 1$.