18.315 PROBLEM SET 2 (due Thursday, October 12, 2006)

1. Let f^λ be the number of standard Young tableaux of shape λ.

(a) (5 points)Prove that the sum $\sum (f^\lambda)^2$ over all partitions λ of n with at most 2 parts (that is $\lambda = (\lambda_1, \lambda_2)$, $\lambda_1 \geq \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = n$) equals the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. (You can use the combinatorial interpretation of C_n as the number of Dyck paths.)

(b) (5 points) Find (and prove) a closed formula for the sum $\sum f^\lambda$ over partitions λ of n with at most 2 parts. The formula might involve a summation.

2. Let $V := \{(z_1, \dots, z_n) \mid z_1 + \cdots + z_n = 0\} \simeq \mathbb{C}^{n-1}$. The symmetric group S_n acts on V by permutations of the coordinates.

(a) (5 points) Find the Gelfand-Tsetlin basis of the representation V.

Hint: Find the basis $v_1, \dots, v_{n-1}$ of V such that each v_i is a common eigenvector of the Jucys-Murphy elements $X_i = (1, i) + (2, i) + \cdots + (i-1, i) \in \mathbb{C}[S_n]$, for $i = 1, \dots, n-1$.

(b) (5 points) Prove that V is equivalent to a certain irreducible representation V_λ of S_n and identify the partition λ.

Hint: Look at eigenvalues of the Jucys-Murphy elements and use the correspondence with content vectors of Young tableaux.

3. (a) (5 points) Prove that the Jucys-Murphy elements X_i and X_j commute with each other (that it $X_i X_j = X_j X_i$) using only the definition of these elements.

(b) (5 points) Let Cyc_n be the element the group algebra $\mathbb{C}[S_n]$ given by $\mathrm{Cyc}_n = \sum w$ over all permutations $w \in S_n$ with a single cycle of size n. Express Cyc_n in terms of the Jucys-Murphy elements for $n = 1, 2, 3, 4$.

(c)* (5 points) Express Cyc_n in terms of the Jucys-Murphy elements for an arbitrary n.

(d)* (5 points) It is clear that $X_1 = 0$, $X_2^2 = 1$. Check that $X_3^3 = 3X_3 + 2X_2$. For any i, express some power $(X_i)^d$ as a polynomial in $f(X_1, \dots, X_i)$ of degree $\deg f < d$.

4. (10 points) Let T be a rooted tree on n nodes. Prove the following "baby hooklength formula:"

$$ext(T) = \frac{n!}{\prod_{v \in T} h(v)}.$$

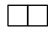
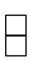
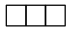
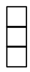
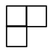
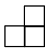
Here $ext(T)$ is the number of *linear extensions* of T, that is $ext(T)$ is the number of ways to label the nodes of T by $1, \dots, n$ so that, for each node labeled i, all children of this node have labels greater than i. The "hooklength" $h(v)$ of a node v in T is the total number of descendants of v (including the node v itself).

5. (a) (5 points) An *involution* is a permutation $w \in S_n$ such that $w^2 = 1$ (that is w has only cycles of sizes 1 or 2). Prove that the number of involutions in S_n equals

$$I_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k\, k!\, (n-2k)!}.$$

(b) (5 points) We know that $\sum_{|\lambda|=n} (f^\lambda)^2 = n!$. Prove that the sum $\sum_{|\lambda|=n} f^\lambda$ equals the number I_n of involutions $w \in S_n$.

6. A *skew Young diagram* $\kappa = \lambda/\mu$ is the set-theoretic difference of two usual Young diagrams shapes λ and μ. For example, $\lambda/1$ is the Young diagram of shape λ with the top left box removed. One can define standard Young tableaux for skew shapes in the usual way as fillings of boxes with numbers $1, \dots, n$ that increase in rows and columns. Let f^κ be the number of such skew Young tableaux.

A *ribbon* is a skew Young diagram such that that (i) it has a single connected component, and (ii) it contains no 2×2-box inside. (We consider ribbons up to parallel translations.) For example, there are 2 ribbons with 2 boxes: ,  ; 4 ribbons with 3 boxes: ,  ,  ,  , etc.

(a) (5 points) Find the number of ribbons with n boxes.

(b) (5 points) Find the sum $\sum f^\kappa$, where κ varies over all ribbons with n boxes.

(c)* (10 points) For given n, find a ribbon κ with n boxes such that f^κ has the maximal possible value (among all ribbons with n boxes). Prove that this is the maximal possible value.

7. A *horizontal k-strip* is a skew Young shape with k boxes that contains no two boxes in the same column. (It may contain several connected components.)

Let U_k and D_l be the operators that act on the space $\mathbb{C}^Y$ of linear combinations of Young diagrams, as follows. $U_k : \lambda \mapsto \sum \mu$, there the sum is over all μ obtained from λ by adding a horizontal k-strip. $D_l : \lambda \mapsto \sum \mu$ there the sum is over all μ obtained from λ by removing a horizontal l-strip. In particular, U_1 and D_1 are the "up" and "down" operators for the Young lattice.

(a) (10 points) Prove that, for any $k, l \geq 0$,

$$U_k U_l = U_l U_k, \qquad D_k D_l = D_l D_k$$

$$D_k U_l = \sum_{r=0}^{\min(k,l)} U_{l-r} D_{k-r}$$

(b)* (10 points) Use these operations to give an alternative proof of the fact that the number of pairs (P, Q) of semi-standard Young tableaux of the same shape and with weights $(\beta_1, \beta_2, \dots)$ and $(\gamma_1, \gamma_2, \dots)$ equals the number of matrices $A = (a_{ij})$ with nonnegative integer entries, with row sums $\sum_j a_{ij} = \beta_i$ and column sums $\sum_i a_{ij} = \gamma_j$ (as in RSK-correspondence).

8. Fix two sequences of integers $r_1, \dots, r_n$ and $c_1, \dots, c_n$. Let S_1 be the set of nonegative integer $n \times n$-matrices $A = (a_{ij})$ with given row sums $\sum_j a_{ij} = c_i$ and column sums $\sum_i a_{ij} = r_j$. Let S_2 be the set of nonnegative integer $n \times n$-matrices $B = (b_{ij})$ such the entries weakly decrease in the rows and in the columns (that is $b_{ij} \geq b_{i',j'}$ whenever $i \leq i'$ and $j \leq j'$) and the diagonal sums $d_k = \sum_{j-i=k} b_{ij}$ are equal to $d_{n-i} = r_1 + \cdots + r_i$ and $d_{-n+i} = c_1 + \cdots + c_i$, for $i = 1, \dots, n$.

(a) (5 points) Construct an explicit bijection between S_1 and S_2 for $n = 2, 3$.

(b) (5 points) Prove that $|S_1| = |S_2|$, for any n.

9*. (10 points) In class we constructed the tranformations of semi-standard Young tableaux $\tilde{s}_i : T \mapsto \tilde{T}$ such that (1) T and $\tilde{T}$ have the same shape, and (2) if the weight of T is $(\beta_1, \dots, \beta_i, \beta_{i+1}, \dots)$ then the weight of $\tilde{T}$ is $(\beta_1, \dots, \beta_{i+1}, \beta_i, \dots)$. Modify these operations and define new operations s_i acting on semi-standard tableaux that satisfy the above properties and, in addition, satisfy the *Coxeter relations:* $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $s_i^2 = 1$, $s_i s_j = s_j s_i$ for $j \neq i \pm 1$.

Then these operations can be extended to the action of the symmetric group on semi-standard tableaux by setting $w(T) := s_{i_1} \cdots s_{i_l}(T)$ for a permutation $w = s_{i_1} \cdots s_{i_l}$.