18.315 Combinatorial Theory 09/07/06

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1. Review basics of repr. theory
2. Reduce algebra to combinatorics
3. Study this combinatorics

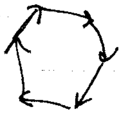
Representation theory

Group ⟷ symmetries of some geom. object

symmetric gp S_n ⟷ finite set ← main discrete gp

gen. lin. gp. GL_N ⟷ linear space ← main continuous gp.

cyclic group. ⟷



circle

Abstract gp G : represent this gp as symmetries of something.

Def. G gp. A (complex, finite dimensional) linear representation is a homomorphism $R: G \to \underset{\overset{\shortparallel}{GL_n(\mathbb{C})}}{GL(V)}$, V - fin. dim. $\underbrace{\text{lin. space}}_{\text{vect sp.}}$

$g \in G$, $\overset{\text{matrix}}{R(g)}$

$R(g_1 g_2) = R(g_1) R(g_2)$, $R(g^{-1}) = R(g)^{-1}$

Example. Cyclic group $C_n = \mathbb{Z}/n\mathbb{Z}$

$$C_n = \{1, g, g^2, \dots, g^{n-1}\} \quad , g^n = 1$$

1-dim'l representations of C_n

$R: C_n \to GL_1 = \mathbb{C} \setminus \{0\}$

$R(g) = z \neq 0$, $R(g^2) = z^2, \dots$ $R(g^n) = z^n = 1$ $\Rightarrow$ z is n-th root of un

There are n different 1-dim'l representations of C_n.

In fact: for any abelian gp # 1-dim reps = order of the group

d-dim reps of C_n:

$R(g) = A \leftarrow d \times d$ matrix. Only condition: $A^n = Id$

Every matrix can be reduced to the Jordan form (by choosing basis suitably,

$$A = \begin{pmatrix} \times & & & & \\ & \times & & & \\ & & \times & & \\ & & & \times & \times \end{pmatrix}$$

$A^n ? =$

but then → it has to be diagonal to get Id.

So $\begin{matrix} \times & & & \\ & \times & & \\ & & \times & \\ & & & \times \end{matrix}$ $\checkmark\checkmark\checkmark$ roots of unity $\Rightarrow$ R "breaks into" 1-dim reps.

Equivalent reps.

matrix is a linear operator.

$A \qquad CAC^{-1}$ same operator, just in 2 different bases.

Two representations R_1, R_2 of G are <u>equivalent</u> if
$R_1(g) = C R_2(g) C^{-1}$ for $\forall g \in G$, some fixed matrix C.

<u>Operations on representations</u>

$R_1 : G \to GL(V)$, $R_2 : G \to GL(W)$ V, W - lin.sp.

1) direct sum $R_3 = R_1 \oplus R_2 : G \to GL(V \oplus W)$

in matrix notation:
$$R_3(g) = \begin{array}{c|c} k & \left(\begin{array}{c|c} R_1(g) & 0 \\ \hline 0 & R_2(g) \end{array}\right) \\ l & \\ & \begin{array}{cc} k & l \end{array} \end{array}$$

2) tensor product $R_4 = R_1 \otimes R_2 : G \to GL(\underset{\uparrow}{V \otimes W})$

multiply dimensions.

3) for any operations on lin. spaces

are similar ops on reps!!! say symm powers, ...

Def. 1) A representation is called <u>decomposable</u> if $R \underset{\text{equiv}}{\overset{\uparrow}{=}} R_1 \oplus R_2$, both

R_1, R_2 are non-trivial , $\dim R_1, R_2 \geq 1$.

In other words, can change basis so that
$$R(g) = \begin{array}{ccc} & k & \\ k & \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right) & l \\ & l & \end{array}$$

Otherwise, R is <u>indecomposable</u>.

2) A rep R is <u>reducible</u> if there exists a subspace $W \subsetneqq V$, $W \neq \{0\}$

$R : G \to GL(V)$

s.t. all operators $R(g)$ preserve W. ⤳ $\forall w \in W,\ R(g).w \in W$

in matrix notation: $$R(g) = \begin{array}{c} \\ k \\ l \end{array}\!\!\begin{array}{c} \begin{array}{cc} k & l \end{array} \\ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \end{array}$$

Otherwise, R is <u>irreducible</u>.

Clearly, irred. is stronger than indecomp:

R irred $\Rightarrow$ R indecomposable

<u>Maschke's Theorem</u>. $\forall$ finite group G, R is irreducible $\Leftrightarrow R$ indecomposable.

i.e. $R = R^{(1)} \oplus R^{(2)} \oplus \cdots \oplus R^{(k)}$, $R^{(i)}$ irreducible.

Moreover, modulo equiv. of rep, these are same in any $R = \tilde{R}^{(1)} \oplus -$

- $\forall$ finite gp G

  - all components ~~de~~$^{(i)}$ are uniquely determined up to isomorphism.
  - there are finitely many irred reps (=irreps)

<u>Questions of Rep. Theory</u>.

) Classify (construct) irreps of G ⟵ $R_1, \ldots, R_\ell$

) $R_i \otimes R_j = \underbrace{R_1 \oplus R_1 \oplus \cdots \oplus R_1}_{m_1} \oplus \underbrace{R_2 \oplus \cdots \oplus R_2}_{m_2} \oplus \cdots = R_1^{\oplus m_1} \oplus R_2^{\oplus m_2} \oplus \cdots$

↑ no longer irred

how to calculate m_k?

$m_{i,j,k}$?

↙

<u>$R_i \otimes R_j$</u>

Symmetric group S_n

elements are bijections $w: \{1,..,n\} \to \{1,2,..,n\}$

one line notation $w(1)\, w(2) \,..\, w(n)$

two line notation $\begin{pmatrix} 1 & 2 & .. & n \\ w(1) & w(2) & & w(n) \end{pmatrix}$

cycle notation $w = (a_1\, a_2 .. a_k)(b_1\, b_2 .. b_\ell)\dots$

multiply $u = 213 = (12)$
$w = 132 = (23)$ $\to u \cdot w =$

$\uparrow$ multiply them as maps

$w: \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$u: \cancel{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}}\ (12)$

$u = \begin{pmatrix} 1\,3\,2 \\ \boxed{2\,3\,1} \end{pmatrix}$ $u \cdot w = (1\,2\,3)$

me reps of S_n

) trivial rep $w \mapsto 1$

) sign rep $w \mapsto \text{sign}(w)$

Exercise. check that these are all 1-dim reps of S_n.

3) defining representation

$R: w \mapsto$ permutation matrix of w

$$R(w)\ \overset{\text{basis elt}}{\overset{\downarrow}{e_i}} \mapsto e_{w(i)} \qquad \forall\, i=1,..,n$$

$$R(312) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

If you take vector $\begin{pmatrix}1\\1\\1\end{pmatrix}^T$ → no perm matrix changes this vector

⇒ def. rep is not irred.

It is not irreducible.

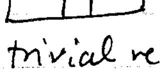
$$\mathbb{C}^n = \underset{\substack{\uparrow \\ \text{spanned by} \\ (1,\dots,1)}}{\{(x,x,\dots,x)\}} \oplus \underset{\substack{\uparrow \\ \text{orthogonal} \\ \text{hyperplane}}}{\{(x_1,..,x_n) \mid x_1+\dots+x_n=0\}}$$

$$R = \underset{\substack{\nearrow \\ \text{trivial rep}}}{R_1} \oplus \underset{\substack{\nwarrow \\ \underline{n-1 \text{ dim'l rep}}}}{R_2}$$

Exercise. Check that R_2 is irreducible.

Irreducible representations of S_n correspond to partitions of n.

S_3

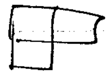


trivial rep

1-dim

sign rep

1-dim



2-dim (subrep) component in the defining rep

2-dim

V_λ irreps of S_n

$\dim(V_\lambda) = f_\lambda$ = #standard Young tableaux of shape λ

$1^2 + 2^2 + 1^2 = 6$

Frobenius-Young identity $\sum_{\lambda \vdash n} (f_\lambda)^2 = n!$

Bernside identity ∀ finite group G

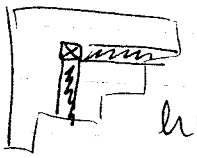
Let $d_1, \ldots, d_\ell$ dimensions of irreps of G, $d_1^2 + \cdots + d_\ell^2 = |G|$

<u>Combinatorially</u>: $\sum_{\lambda \vdash n} (f_\lambda)^2 = n!$

$\{(P, Q) \mid 2 \text{ tableaux of same shape } \lambda \vdash n\} \leftrightarrow S_n$

Schensted correspondence

<u>Hook-length formula</u>: $f_\lambda = \dfrac{n!}{\prod_{x \in \lambda} h(x)}$



$h(x)$ = # boxes in this hook.

$\lambda = (n, n) \rightarrow f_\lambda = \dfrac{(2n)!}{(n!)\, 2 \cdot 3 \cdots (n+1)} = \dfrac{1}{n+1}\binom{2n}{n} \Leftarrow$ Catalan numbe

<u>Example</u>. S_n acts on $\mathbb{C}[x_1, \ldots, x_n]$

$w : f(x_1, \ldots, x_n) \mapsto f(x_{w(1)}, \ldots, x_{w(n)})$

P_k = space of polynomials of degree k

$P = P_0 \oplus P_1 \oplus P_2 \oplus \cdots \leftarrow$ symmetric powers of $\overset{P_1}{\downarrow}$ defining reps

$\uparrow$ perm on const term

$\uparrow$ perm on linear terms → n-dim rep → defining rep

perms don't change const term

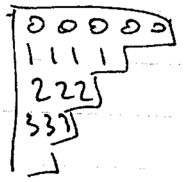
⇒ trivial rep

$$P_k = \bigoplus_{\lambda \vdash n} V_\lambda^{\oplus m_{\lambda,k}}$$ ← multiplicity of V_λ

$$g_\lambda(t) = \sum_{k \geq 0} m_{\lambda,k} \cdot t^k$$

Theorem.

$$g_\lambda(t) = \frac{t^{n(\lambda)}}{\prod_{x \in \lambda} (1 - t^{h(x)})} \quad , \quad n(\lambda) = \sum \lambda_i (i-1)$$



sum these numbers

$\lambda = (n)$

$V_{(n)}$ trivial rep

symmetric polynomials $f(x_1,\dots,x_n) = f(x_{w(1)},\dots,x_{w(n)}) \quad \forall w \in S_n$

$\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \dots$

$m_{(n),k} = \dim(\Lambda_k)$

↑ space of symm polys of degree k

$$g_{(n)} = \frac{1}{(1-t)(1-t^2)\cdots(1-t^n)}$$

$\lambda = (1,1,\dots 1) = (1^n)$

we are looking antisymm polys

Office hours: TR 4-5

09/12/06

Problem sets - roughly every 2 weeks

G - finite group
$H \subset G$ subgroup
V representation of G

$\mathrm{Res}_H^G V$ - restriction of rep V to subgroup H

very important - especially in S_n

if V is irred → restr. is not necessarily irred

Facts. $G = S_n$

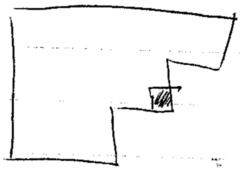
irreps V_λ, λ - partition of n

$\mathrm{Res}_{S_{n-1}}^{S_n} V_\lambda$, need to imbed S_{n-1} into S_n
$S_{n-1} \hookrightarrow S_n$ in standard way;
acts on first $n-1$ letters

Branching rule

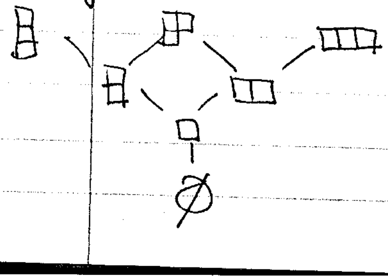
$$\mathrm{Res}_{S_{n-1}}^{S_n} V_\lambda = \bigoplus_{\mu : \mu \nearrow \lambda} V_\mu$$

Young diagram of μ is obtained from λ by removing 1 corner-box



think abt this rule combinatorially

Young lattice $\mathbb{Y}$



$$\underset{\text{vect sp}}{V_\lambda} = \underset{\text{sum of other vect sp.}}{\bigoplus V_\mu}$$

Corollary. $\dim V_\lambda = \# SYT(\lambda)$

Pf. $\dim V_\lambda = \sum_{\mu : \mu \nearrow \lambda} \dim V_\mu \underset{\substack{\text{restrict}\\ \text{to } S_{n-2}}}{=} \sum_{\nu \nearrow \mu \nearrow \lambda} \dim V_\nu = \dots =$

$$= \sum_{\emptyset = \lambda^{(0)} \nearrow \lambda^{(1)} \nearrow \dots \nearrow \lambda^{(n)} = \lambda} \dim V_\emptyset = \#\text{ paths from } \emptyset \text{ to } \lambda \text{ in } \mathbb{Y}.$$

S_0 - $0!$ elts
↳ empty product $= 1$

$n! \to$ prod of n elts

$S_0 = \{1\}$
→ 1 1-dim triv repr
$\dim V_\emptyset = 1$

$$\emptyset \nearrow \boxed{1} \nearrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \nearrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \nearrow \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & & \\ \cline{1-1}\end{array}$$

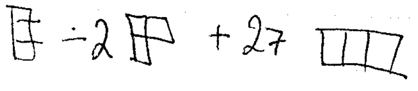
w/ 1-1 corresp w/ SYTs.

Theorem $\sum_{\lambda \vdash n} f_\lambda^2 = n!$

Pf. Prove this statement combinatorially

LHS = # paths in $\mathbb{Y}$ that start & end at $\emptyset$ and have n up steps followed by n down steps.

$\mathbb{C}^{\mathbb{Y}}$ - space of all (finite) formal linear combinations of Young diagrams

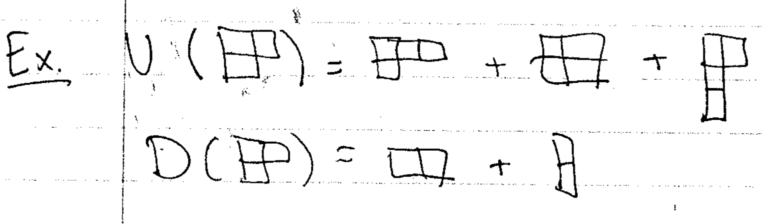


$\mathbb{C}^{\mathbb{Y}}$ - ∞-dim'l vect sp.
$\emptyset$ is also basis elt

- Define up & down operators

$$U : \lambda \mapsto \sum_{\mu : \lambda \nearrow \mu} \mu$$

$$D : \cancel{\lambda} \mapsto \sum_{\mu : \mu \nearrow \lambda} \mu$$

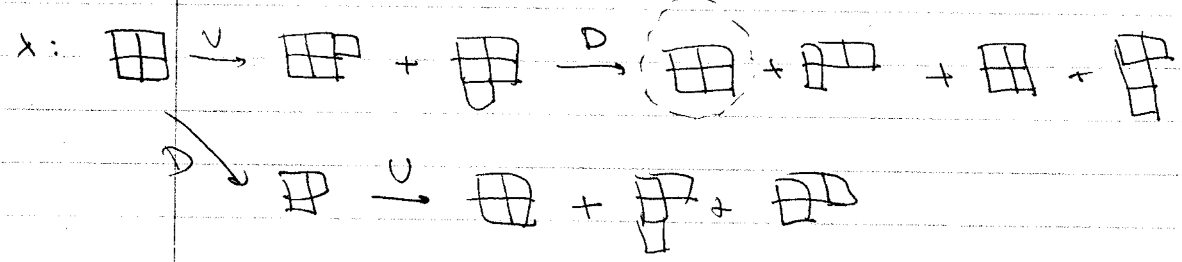


$D^n U^n \emptyset = n! \emptyset$ ← theorem says

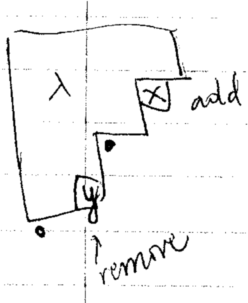
basis elt

Lemma. (key lemma) $D \cdot U = U \cdot D + 1$ ↙ identity operator

(Heisenberg algebra)



$DU(x)$



if $x \neq y$ can do $UD(\lambda)$

so UD same as DU

except if $x = y$ → then get I

[illegible] it for DU - #inner corners

UD - $x = y$ #outer corners

#inner corner = #outer corners-1

$\Rightarrow$ $\boxed{DU - UD = 1}$

1) $DU = UD + 1$

2) $D \cdot \emptyset = 0$ - nothing below $\emptyset$

$$\underbrace{DD\cdots D}_{n}\ \underbrace{U\cdots U}_{n}\ \emptyset$$

↑

get a matching between D's & U's
bec we want D to annihilate U

↓

matchings $\underline{n!}$

$$\cdots\ DU \cdots \;=\; \cdots UD \cdots \;+\; \cdots\ \cdots$$

D cannot go all the way down to --- $D\emptyset$, bec that is 0.

D, U - operators, so they are associative

Another proof.

space: $\mathbb{C}[x]$

$U \longleftrightarrow x$ operator of mult. by x $\quad f(x) \mapsto x \cdot f(x)$

$D \longleftrightarrow \frac{d}{dx}$ $\quad f(x) \mapsto f'(x)$

$\emptyset \longleftrightarrow 1$

we get same relation $DU = UD + 1$ - Product rule for $\cdot x$, $\frac{d}{dx}$

So in order to prove $D^n U^n \emptyset = n!\, \emptyset$

$$\left(\frac{d}{dx}\right)^n x^n \; 1 = n!$$

Exercise. Think of $\mathbb{Y}$ as undirected graph

\# paths in $\mathbb{Y}$ from $\emptyset$ to $\emptyset$ with $2n$ steps $= (2n-1)!! = 1\cdot 3\cdot 5 \cdots (2n-1)$

Back to basics of representation theory

Group algebra   e.g. $\mathbb{C}[x]$ is an algebra

usually over $\mathbb{C}$ alg, over $\mathbb{Z}$ ring

Def. $G = \{g_1, \dots, g_k\}$ finite group.

$\mathbb{C}[G]$ = group algebra.

- as a vector space, elt are formal linear comb

$$\alpha_1 g_1 + \alpha_2 g_2 + \cdots + \alpha_r g_r, \quad \alpha_1, \dots, \alpha_r \in \mathbb{C}$$

(as a vect sp it is isomorphic to $\cong \mathbb{C}^r$)

- multiplication is given by

$$(\alpha_1 g_1 + + \alpha_k g_k)(\beta_1 g_1 + \cdots + \beta_k g_k) = \sum_{i,j} (\alpha_i \beta_j)\,\overset{\text{product in } G}{(g_i \cdot g_j)}$$

simple, but very nontrivial!

Example $G = S_3$ $\underset{\text{as vect sp 6-dim'l}}{\mathbb{C}[S_3]} = \{\alpha \cdot \overset{\text{identity}}{1} + \beta \cdot (1,2) + \gamma(1,3) + \delta(2,3) + \cdots\}$

$$(1 + \underset{\text{cycle notation}}{(1,2)}) \cdot ((1,2) - 2(2,3)) = (1,2) - 2(2,3) + 1 - 2(1,2,3$$

Ex. $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[x, x^{-1}]$ ← ring of polynomial

Defn G-module = (left) module over $\mathbb{C}[G]$

vector space V with operation $g \in G, v \in V$, $g \cdot v \in V$

G-module is exactly same as representation
$\mathbb{C}[G]$-module

Def. V G-module, G-submodule $W \subset V$ s.t. $\forall w \in W, g \in G$, $g \cdot w \in$

same as subrepresentation

Def. regular representation $V = \mathbb{C}[G]$ group acts on this space by left multiplication

$$g \cdot (\alpha_1 g_1 + + \alpha_k g_k) = \alpha_1 (g g_1) + + \alpha_k (g g_k)$$

$\dim V = \# G$.

every irred repr is contained in V.

Theorem. Let ~~$\mathbb{C}[G]$e~~ Let V_i, $i \in I$ are all irreps of G, G finite

$$\underset{\text{(some people denote by R)}}{\overset{\text{regular rep}}{\to}} \mathbb{C}[G] = \bigoplus_{i \in I} (\dim V_i) V_i$$

$$\sum_{i \in I} (\dim V_i)^2 = |G|$$ (Burnside identity?)

Ex. $G = S_3$

$$\mathbb{C}[S_3] = V_{\ydiagram{3}} \oplus V_{\ydiagram{2,1}} \oplus V_{\ydiagram{2,1}} \oplus V_{\ydiagram{1,1,1}}$$

trivial rep ↗ ↑ ↗ 2 dim rep ↗ sign rep

How to find $V_{\ydiagram{3}}$ in $\mathbb{C}[S_3]$

only vector satisfying (this)? property is $\sum_{w \in S_3} w$

$$V_{\ydiagram{3}} \to \langle \sum_{w \in S_3} w \rangle$$

mult. by even perm stays same
odd " changes sign

$$V_{\ydiagram{1,1,1}} \to \overset{\text{span}}{\langle} \sum_{w \in S_3} \underset{\uparrow \text{ sign } w}{(-1)^w} w$$

How do we find $V_{\ydiagram{2,1}} \oplus V_{\ydiagram{2,1}}$ inside $\mathbb{C}[S_3]$?

↖ ↗ 2 dim reps

Consider elt $c = (1+(1,2))\cdot(1-(1,3)) = 1 - (1,3) + (1,2) - (1,3,2) \in \mathbb{C}$

↙ consider submodule which this vector generates

Take $\mathbb{C}[S_3]\cdot c \subset \mathbb{C}[S_3]$

↑ subspace of $\mathbb{C}[S_3]$, submodule, minimal containing c

we claim $V_{\ydiagram{2,1}} \simeq \mathbb{C}[S_3]$ — special case of construction called Young symmetrizer

Young symmetrizer

Pick any tableau T of shape λ

$$T = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}$$

arb. perm of 1,3,4 & of 2,5

$a_T = \big(\underbrace{1 + (1,3) + (1,4) + (3,4) + (1,3,4) + (1,4,}_{\text{on } 1,3,4}$

$\cdot \; (1 + (2,5)\,)\big)$

times

$$a_T = \sum_{\substack{w \in S_n \\ w \text{ preserves rows of } T}} w \in \mathbb{C}[S_n]$$

first column on 1,2   2nd column

$b_T = (1 - (1,2))\,(1 - (3,5))$

nothing on 3-rd column, just identity

$$b_T = \sum_{\substack{w \in S_n \\ w \text{ preserves columns of } T}} (-1)^w \, w$$

$$V_T = \mathbb{C}[S_n] \cdot a_T \cdot b_T \subseteq \mathbb{C}[S_n]$$

Theorem 1) V_T is an irrep.

2) $V_T \simeq V_{T'}$ whenever T & T' have same shape

and $V_T \not\simeq V_{T'}$ if not of ~~the~~ same shape

$\hookrightarrow$ holds for any tableau! not necessarily SYT.

Remark: here we can take any tableau & give irrep

but if we take ~~any~~ SYT – will give decomposition

$$\mathbb{C}[G] = \bigoplus_{i \in I} (\dim V_i)\, V_i$$

Proof of Maschke's Theorem.

G - finite group, V - G-module, $W \subset V$ G-submodule

$\exists W'$ G-submodule, $W' \subset V$ s.t. $V = W \oplus W'$.

Now if W, W' irred, fine, if not $\to$ find another submod & break it all down after get irred.

Want some "orthogonal complement" - need some inner product

good one - invariant w.r.t. action of

Def. An inner product $\langle \cdot, \cdot \rangle$ on V is G-invariant if $\langle x, y \rangle = \langle gx, gy \rangle$, $x, y \in V$, $g \in G$.

Pick any inner product, $\langle \cdot, \cdot \rangle$ e.g. pick arbitrary basis, & let it be othogonal.
Now, make it G-invariant

Define : $\langle x, y \rangle' = \sum_{g \in G} \langle g.x, gy \rangle$, G-invariant

bec if g ranges over all of $G \Rightarrow g.h$ $\xrightarrow{1-1}$ G

only place: here we used that $\underline{G \text{ is finite}}$, as o.w. cannot have this

$W^\perp = \{ v \in V \mid \langle v, w \rangle' = 0 \ \ \forall w \in W \}$. Claim $W^\perp$ is G-submodule.

$\forall v \in W^{\perp}, g \in G$ need $g \cdot v \in W^{\perp}$

$$\langle g \cdot v, w \rangle' = \langle g^{-1} g w, g^{-1} w \rangle' = \langle v, \underset{\substack{\wedge \\ W \text{ as } G\text{-module}}}{g^{-1} w} \rangle' = 0.$$

$\forall w \in W.$

①...

PSET online ② Additional subsection of this problem (U-D-U-U-UD-..) **09/14/06**

match them → in $(2n-1)!!$ ways

can use this in HW.

③ if $YX - XY = Y$ ← commutator

$$Y^m X^n = \sum_{k=0}^{m} \binom{m}{k} n^{m-k} X^n Y^k$$

④ Construct explicitly 2-dim irrep of S_4 ← generated by adjacent transpositions

$s_i = (i, i+1)$

need to present 3 matrices

unique such → later abt it.

⑤* Let F_n = set of all sequences $(a_1, \dots, a_k)$ ~~max~~ $a_i \in \{1, 2\}$ and $a_1 + \dots + a_k = n$. (compositions of n into parts of size 1 and 2)

$F_0 = \{\emptyset\}$ $F_2 = \{11, 2\}$, $F_3 = \{111, 21, 12\}$

$F_1 = \{1\}$ / Make a graph on the set $F = F_0 \cup F_1 \cup F_2 \cup \dots$

edges $(a_1 \dots a_k 2^{\ell}) = (a_1 \dots a_k \underbrace{2 .. 2}_{\ell}) \overset{\text{edge connect}}{-} a_1 .. a_k 1 \underbrace{2 .. 2}_{\ell}$

$(a_1 \dots a_k 1 \underbrace{2 .. 2}_{\ell}) - (a_1 .. a_k \underbrace{2 .. 2}_{\ell+1})$ ← edges ↗

Show that this graph has same property as Young graph.

For $\bar{a} \in F_n$ let $p(\bar{a})$ = # paths from $\emptyset$ to $\bar{a}$ (w/ n edges)

Prove that $\sum_{\bar{a} \in F_n} p(\bar{a})^2 = n!$.

in 1 week, PSET due.

Claim. $[D, U] = I$ ← commutator, $D \cdot \emptyset = 0$. Then $D^n U^n \emptyset = n! \emptyset$

Proof. $D U^n \emptyset = (UD + I) U^{n-1} \emptyset = U D U^{n-1} \emptyset + U^{n-1} \emptyset =$

$DU = UD + I$

$$= U(UD+I)U^{n-2}\emptyset + U^{n-1}\emptyset =$$

$$= U^2 D U^{n-2} \emptyset + 2U^{n-1}\emptyset = \ldots =$$

$$= \underline{U^n \cancel{D\emptyset}} + n U^{n-1} \emptyset$$

now multiply by D again : $D^2 U^n \emptyset = n(n-1) U^{n-2} \emptyset$

$D^3 U^n \emptyset = n(n-1)(n-2) U^{n-3} \emptyset$

etc : $D^n U^n \emptyset = n! \emptyset$

Classical results from representation theory:

Schur's lemma.

First definition: V, W G-modules, then a G-homomorphism, (also called intertwining operator) $A : V \to W$ is s.t. $A(gv) = g A(v)$ for any $g \in G$, $v \in V$. (equivalently if $A : v \to w$ then $A : gv \mapsto gw$)

i.e. operator A respects the action of the group / commutes w/ the — " —

G-isomorphism is an invertible G-homomorphism.

If V, W are G-isom ⇒ they are equiv, they are same in some basis.

$\mathrm{Hom}_G(V, W)$ = space of G-homomorphisms

Schur's lemma. If V, W are irreducible and A is a G-homomorphism, then either $A = 0$ or A is invertible.

Proof. $\ker A \subset V$, $\operatorname{im} A \subset W$

Fact that A is G-hom ⇒ $\ker A$, $\operatorname{im} A$ should be, and is, a G-submo

Since V, W irred ⇒ $\ker A$, $\operatorname{im} A$ are either 0 or the whole space

$$\ker A = V, \quad \Rightarrow A = 0$$
$$\ker A = 0 \quad \Rightarrow \operatorname{im} A = W.$$ ▨

Assume that $V = W$, map $(A - cI) : V \to V$ also G-hom, if A is.

Now pick c to be an eigenvalue of A, so $A - cI$ cannot be invertible ⇒ by Schur lemma it is 0.

Corollary. Any G-hom from V to V, V irred is a multiple of I, i.e. $A = c$

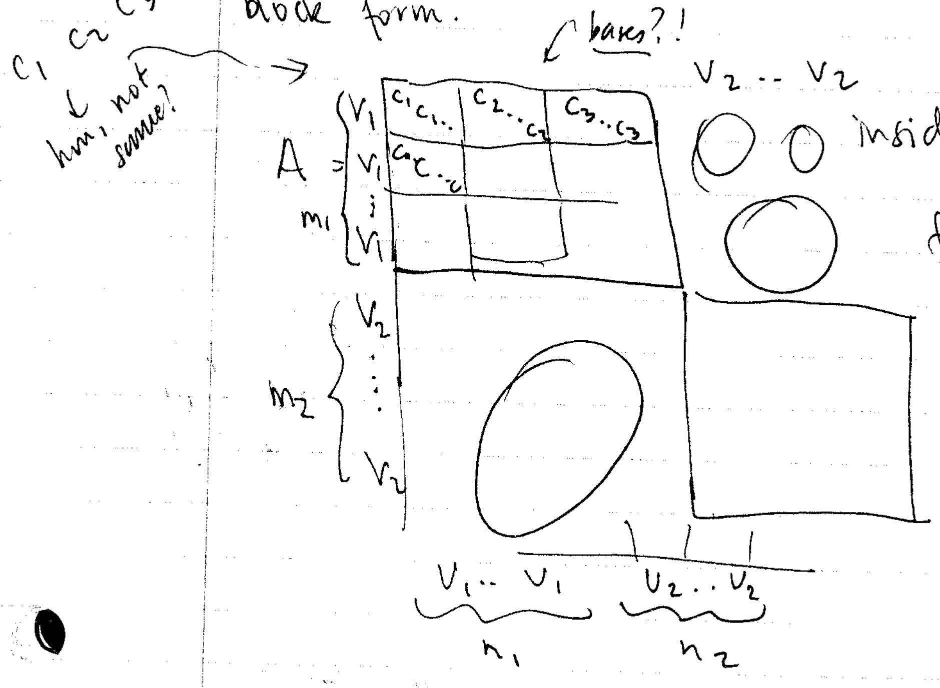
⇒ in this case space of these homom's is a 1-dim'l space.

Let $V_1, .., V_k$ be all irreps of G (haven't proved yet that there are finitely many of these

Then $V = m_1 V_1 \oplus .. \oplus m_k V_k$, V any representation.

Let $W = n_1 V_1 \oplus .. \oplus n_k V_k$

Pick ~~any~~ those bases of V and W that agree with these decompositions, and in these bases a G-hom A will have block form.



inside each block we have a G-homom from V_j to V_i

now from V_i to V_i

$\begin{bmatrix} c & \ddots & 0 \\ 0 & & c \end{bmatrix}$ by

Schur.

$\begin{array}{c|c} & V_i \\ \hline V_j & 0 \end{array}$ bec

there's no G-hom between V_i & V_j

Cor $\dim \mathrm{Hom}_G(V,W) = m_1 n_1 + m_2 n_2 + \dots + m_k n_k$
"intertwining number"

Cor. So, if $V = n_1 V_1 \oplus n_2 V_2 \oplus \cdots \oplus n_k V_k \overset{\text{isom.}}{\simeq} n_1' V_1 \oplus \cdots \oplus n_k' V_k$

then $n_i = n_i'$.

Proof. $n_i = \dim \mathrm{Hom}_G(V, V_i)$ by previous formula.

Characters. Let $R: G \to GL(V)$ any representation, finite dim'l.

character of R is $\chi_R : G \to \mathbb{C}$

$$g \mapsto \operatorname{tr} \underbrace{R(g)}_{\text{operator (matrix)}}$$

$\chi_V = \chi_R$.

Slight difference between repr. & modules
$\boxed{\text{in notation}}$ $\quad V \quad\quad R$

Conjugacy classes: $g_1 \sim g_2$ if $\exists\, h \in G$ s.t. $g_1 = h g_2 h^{-1}$

equivalence relation.

Key property of characters: χ_R is $\underbrace{\text{constant on conjugacy classes}}_{\text{such fncts are called class functions}}$.

Proof.:

$$\operatorname{tr}(BAB^{-1}) = \operatorname{tr} A \quad \blacksquare$$

xample. R - regular representation

$G = \{g_1, \dots, g_n\}$, reg. rep. acts on n-dim space, given by matrix $(R(g))_{ij} = \begin{cases} 1 & \text{if } g \cdot g_i = g_j \\ 0 & \text{o.w.} \end{cases}$ $\quad g \cdot g_i = g$

$$\chi_R(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{o.w.} \end{cases}$$

ore elementary properties of characters

1) $\chi_R(1) = \dim R$ ← repr.

2) $\chi_{R_1 \oplus R_2} = \chi_{R_1} + \chi_{R_2}$ $\quad\quad R_1 \oplus R_2 = \left(\begin{array}{c|c} R_1(g) & 0 \\ \hline 0 & R_2(g) \end{array}\right)$

3) $\chi_{R_1 \otimes R_2} = \chi_{R_1} \cdot \chi_{R_2}$

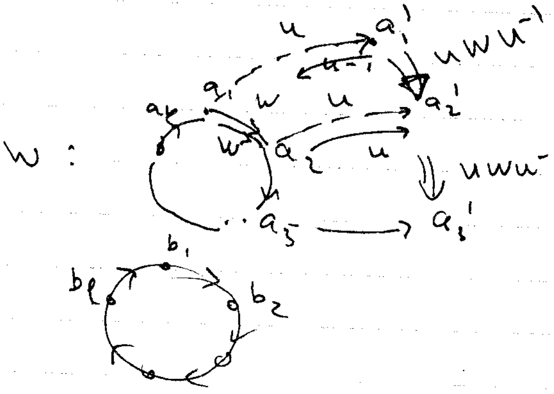
$$\chi_{R_1 \otimes R_2} = \chi_{R_1} \chi_{R_2}$$

$$\mathrm{tr}\,(A \otimes B) = \mathrm{tr} \left(\begin{array}{c|c|c} a_{11} B & a_{12} B & \cdots \\ \hline a_{21} B & a_{22} B & \cdots \\ \hline \cdots & \cdots & \ddots \end{array} \right) =$$

$$= a_{11}\,\mathrm{tr}B + a_{22}\,\mathrm{tr}B + \cdots \quad = \mathrm{tr}A \cdot \mathrm{tr}B .$$

Somehow, it is better to do everything w/o choosing a basis – but we are trying to be more explicit in this class.

Example. $G = S_n$



$$w = (a_1 \ldots a_k)(b_1 \ldots b_\ell) \ldots$$

$$uwu^{-1} = (a_1', \ldots, a_k')(b_1' \ldots b_\ell') \ldots$$

So it has same cycle structure
but since we can pick any u, we can shift numbers in any way!

cycle(w) = cycle partition of w is a partition of n into lengths of cycles in w

Claim $u \sim w$ iff cycle(u) = cycle(w).

Showed above.

Character table for S_3

	1 elt id. 1	3 elts (1,2)	2 elts (1 2 3)
trivial rep	1	1	1
sign rep	1	−1	1
2-dim'l irrep	2	0	−1

← conj. classes

since value at id is = dim of rep

calculating this entry:

S_3 acts by permutation of coordinates $\{(x,y,z) \mid x+y+z=0\} \simeq \mathbb{C}^2$

if S_3 has only 1 irred. 2-dim'l rep → then it should be this representation

(1,2) → switches x and y

$R_{(12)}: (x,y) \mapsto (y,x)$ given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

of trace 0

$R_{((123))}: (x,y) \mapsto (y,-x,-y)$ given by $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

or

$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

trace −1 ←

Hermitian inner product of characters $\chi, \Psi : G \to \mathbb{C}$

$$\langle \chi, \Psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \cdot \overline{\Psi(g)}$$

↑ since complex fncts.

Instead of $\overline{\Psi(g)}$ we can write ~~[illegible]~~ $\Psi(g^{-1})$.

We said there's a G-invariant inner product → so can pick a bais

so that all matrices are unitary

One can pick a basis in (~~a G-module~~) repr. $R : G \to GL(V)$ s.t.

$R(g)$ are unitary. This means that $R(g^{-1}) = (R(g))^{-1} = \overline{R(g)^T}$

Lemma $\overline{\chi(g)} = \chi(g^{-1})$.

var

Theorem. χ_V, χ_W chars of irreps. Then $\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & \text{if} \\ 0 & \text{o.} \end{cases}$

So, characters are orthogonal to each other.

Proof. $R_1 : G \to GL(V)$, $R_2 : G \to GL(W)$

Hom(V, W) – all linear homomorphisms (not nec G-invariant)
$A : V \to W$

Let $R_3(g)$ act on Hom(V,W) by

$$R_3(g) : A \longmapsto R_2(g)\, A\, R_1(g^{-1})$$

homom A ~~matrix A~~

first acts on columns of this matrix
second – " rows – "

● <u>Claim:</u> $\chi_{\mathrm{Hom}(V,W)}(g) = \chi_V(g^{-1}) \cdot \chi_W(g)$

<u>Short proof:</u> $\mathrm{Hom}(V,W) = W \otimes V^* \rightarrow \underline{\overset{\text{space of}}{f: V \to \mathbb{C}}}$ – dual of V.

and we already know for $\otimes$ tr multipl.

<u>in terms of basis:</u> Pick bases of V, W, then $\mathrm{Hom}(V,W)$ is space of matrices, so

$$R_3(g): \mathrm{Mat}_{m\times n} \longrightarrow \mathrm{Mat}_{m \times n}$$

$\downarrow$ pick basis

$$E_{ij} = \begin{bmatrix} 0 & \\ \overset{1}{\cancel{0}} & 0 \end{bmatrix}$$

(i,j)

$$R_3(g): A \mapsto R_2(g) A R_1(g^{-1})$$

● $$E_{ij} \mapsto \sum_{k,l} (R_2(g))_{ki} (R_1(g^{-1}))_{jl}$$

$$\mathrm{tr}\, R_3(g) = \sum_{i,j} R_2(g)_{ii} R_1(g^{-1})_{jj} = \mathrm{tr}\, R_2(g)\, \mathrm{tr}\, R_1(g^{-1}).$$

$$\varphi: A \longrightarrow \frac{1}{|G|} \sum_{g\in G} R_3(g)\cdot A$$ is a projection from $\mathrm{Hom}(V,W)$ to $\mathrm{Hom}_G(V,W)$

homomorphism $\overset{A}{\vee}$ is G-invariant iff $R_3(g) A = A \quad \forall g \in G$.

lef. $$\mathrm{tr}\,\varphi = \frac{1}{|G|} \sum_{g\in G} \mathrm{tr}\, R_3(g) = \frac{1}{|G|}\sum_{g\in G} \mathrm{tr}\, R_2(g)\, \mathrm{tr}\, R_1(g^{-1}) = \begin{cases} 0 & \text{if } V \not\sim W \\ 1 & \text{if } V \sim W \end{cases}$$

$$\| \quad \frac{1}{|G|}\sum_{g\in G} \chi_W(g)\chi_V(g^{-1})$$

● Apply Schur's lemma

1) if $V \not\sim W$ $\quad \mathrm{tr}\,\varphi = 0$

2) If $V \sim W$ $\quad$ take n-dim space project onto line →

$\mathrm{tr}\,\varphi = 1$.

Last time: characters are orthogonal

09/19/06

So, if we have 2 reducible reps $V = \oplus m_i V_i$, $W = \oplus n_i V_i$

Corollary of orthog. $V = \oplus m_i V_i$, $W = \oplus n_i V_i$

$$\langle \chi_V, \chi_W \rangle = \underset{\uparrow \text{intertwining number}}{\sum m_i n_i} = \dim \mathrm{Hom}_G(V,W)$$

Corollary $V \sim W$ iff $\chi_V = \chi_W$.

Example. $R = \mathbb{C}[G]$ regular representation

$$\langle \chi_R, \underset{\uparrow \text{irrep}}{\chi_{V_i}} \rangle = \frac{1}{|G|} \sum_{g \in G} \underbrace{\chi_R(g)}_{\downarrow} \chi_{V_i}(g^{-1}) = \frac{1}{|G|} \cdot |G| \cdot \dim V$$

0 everywhere except 1
where $\chi_R(1) = |G|$
$\chi_{V_i}(1) = \dim V_i$

$\Rightarrow$ $\langle \chi_R, \chi_{V_i} \rangle = \dim V_i$

Corollary $\mathbb{C}[G] = \bigoplus_i (\dim V_i)\, V_i$

In particular, # nonequivalent irreps $< \infty$ since $\underset{|G|}{\dim \mathbb{C}[G] < \infty}$

and every V_i enters in $\bigoplus_i (\dim V_i)\, V_i$.

Def. $\mathbb{C}_{\text{class}}(G)$ = space of class functions on G

$$= \{ \alpha : G \to \mathbb{C} \mid \alpha(hgh^{-1}) = \alpha(g) \ \forall g, h \in G \}$$

<u>Theorem</u>. Irreducible characters χ_{V_i} form an orthogonal basis in $\mathbb{C}_{\text{class}}(G)$.

<u>Proof</u>: We saw χ_{V_i} orthogonal $\Rightarrow$ linear ind.

We only need to show they <u>span</u> $\mathbb{C}_{\text{class}}(G)$.

Pick any $\alpha \in \mathbb{C}_{\text{class}}(G)$ s.t. $\langle \alpha, \chi_{V_i} \rangle = 0 \quad \forall i$

Need $\alpha = 0$.

$R: G \to GL(V)$ irrep

average repr. matrices w/ $\overset{\text{weights}}{\alpha}$.

$$\varphi_{\alpha,V} = \sum_{g \in G} \alpha(g) R(g^{-1}) : V \to V$$

Then $\varphi_{\alpha,V}$ is a G-homomorphism. Need for this:

$\varphi_{\alpha,V} R(h) = R(h) \varphi_{\alpha,V} \quad \forall h \in G$ ... need to write it out

remember $\alpha(hgh^{-1}) = \alpha$

since $\alpha \in \mathbb{C}_{\text{cla}}$

By Schur lemma $\varphi_{\alpha,V} = cI$. Then $c = \frac{1}{\dim V} \operatorname{tr}(\varphi_{\alpha,V})$

$$= \frac{1}{\dim V} \sum_{g \in G} \alpha(g) \operatorname{tr} R(g^{-1}) =$$

$$\operatorname{tr} R(g^{-1}) = \chi_R(g^{-1}) \to \chi_{\textcircled{V}}(g^{-1})$$

V

$$= \frac{|G|}{\dim V} \langle \alpha, \chi_V \rangle = 0$$

$\Rightarrow c = 0$

$\Rightarrow$ For any irrep $\varphi_{\alpha,V} = 0 \Rightarrow \forall$ rep V, $\varphi_{\alpha,V} = 0$

$\downarrow$

$V = \bigoplus m_i V_i$

In particular, for regular representation R, $\varphi_{\alpha,R} = 0$. But by defn $\varphi_{\alpha,R} = \sum_{g\in G} \alpha(g) R(g^{-1}) = 0$. Apply it to identity 1, $\sum_{g\in G} \alpha(g) g^{-1} \in \mathbb{C}[G]$

and $\sum_{g} \alpha(g) \underset{\uparrow \text{ indep elts}}{g^{-1}} = 0 \Rightarrow \alpha(g) = 0 \ \forall g \in G$.

Corollary. # irreps = # conj. classes, because $\dim \mathbb{C}_{\text{class}}(G)$ = # conj. classes, so we get a basis here.

Remark. In general, there is no "natural" ~~map~~ correspondence between irreps and conjugacy classes. → but for symmetric gp we'll see there is a "natural" corresp through partitions λ

Character table for S_4.

# elts conjugacy classes	1 1	6 (12)	8 (123)	6 (1234)	3 (12)(34)
trivial rep V_1	1	1	1	1	1
sign V_2	1	-1	1	-1	1
3-dim'l rep V_3 subrep of the defining rep	3	1	0	-1	-1
$V_2 \otimes V_3$	3	$-1\cdot 1 = 1$	$1\cdot 0 = 0$	$-1\cdot -1 = 1$	$1\cdot -1 = -1$
	2	a_1 $=$ 0	a_2 $=$ -1	0	$+2$

bec 3-dim'l ←

$1\cdot 1 + 1\cdot(-1) + 3\cdot(1) + 3(1) + 2\cdot a_1 = 0$

$1\cdot 1 + 1\cdot 1 + 3\cdot 0 + 3\cdot 0 + 2\cdot a_2 = 0$

can calculate since we can explicitly construct matrices as last time for S_3

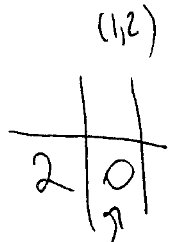
We have regular reps decomposing into irreps. $\mathbb{C}[G] = \oplus(\dim V_i) V_i$

$$\chi_{\mathbb{C}[G]}(g) = \sum \dim V_i \, \chi_{V_i} = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{o.w.} \end{cases}$$

Problem 4 asked to represent 2-dim'l rep as matrices.

Need to construct 2×2 matrices $R(w)$ $\forall\, w \in S_4$.

S_n is generated by adjacent transpositions $s_i = (i, i+1)$



A, B, C

$$\Rightarrow \underset{(12)}{\underset{\uparrow}{\operatorname{tr} A}} = \underset{(23)}{\underset{\uparrow}{\operatorname{tr} B}} = \underset{(34)}{\underset{\uparrow}{\operatorname{tr} C}} = 0$$

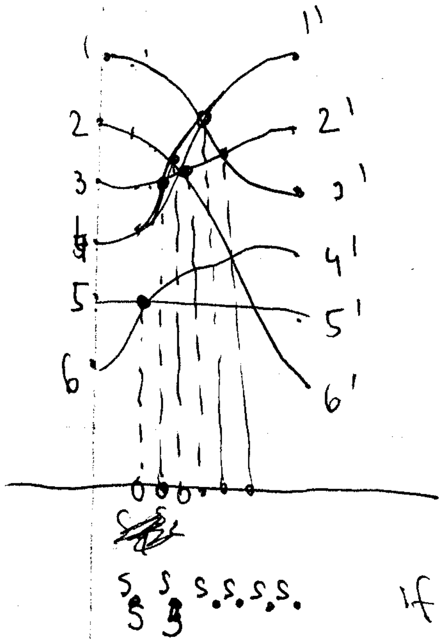
$, A^2 = B^2 = C^2 = I$

$\Rightarrow$ eigenvalues are ± 1

we can diagonalize first 1.

Graphical representation for relations in S_n

<u>Wiring diagrams.</u>



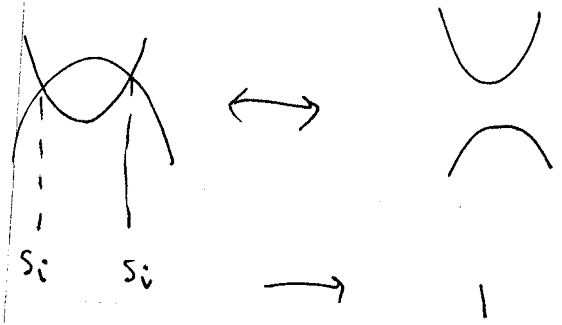
$\forall i$ connect i w/ $w(i)'$

curves general position & no 2 intersection w/ same x coordinates

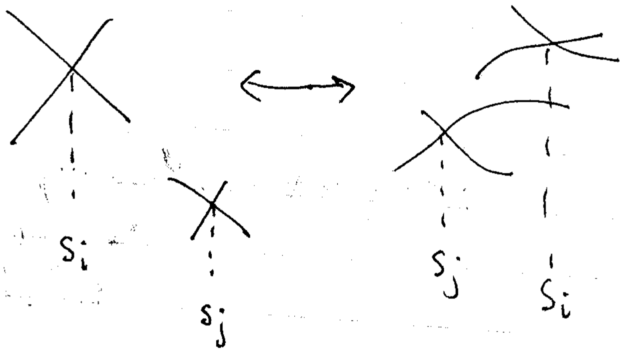
$$w = s_2 s_5 s_4 s_1 \; s_5 s_3 s_2 s_3$$

If we flip this

$s_j \to$ switched jth & $(j+1)$th from the top

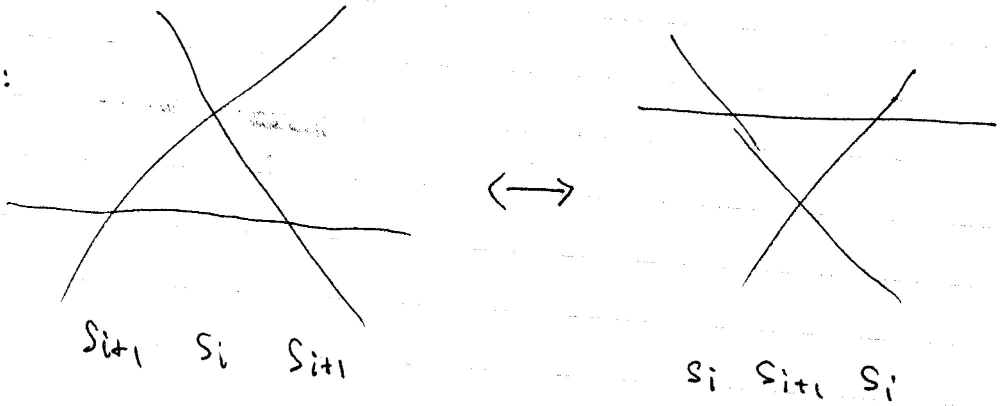


← in these diagrams these things can happen



if $|i-j|>1$

ost important ransformation is:



ll other ~~genera~~ transform. of wiring diagram is gen by these 3 tranform.

<u>Theorem</u>. S_n is generated by $s_1, \dots, s_{n-1}$ with relations:

(1) $s_i^2 = 1$
(2) $s_i s_j = s_j s_i$ $|i-j|>1$
(3) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

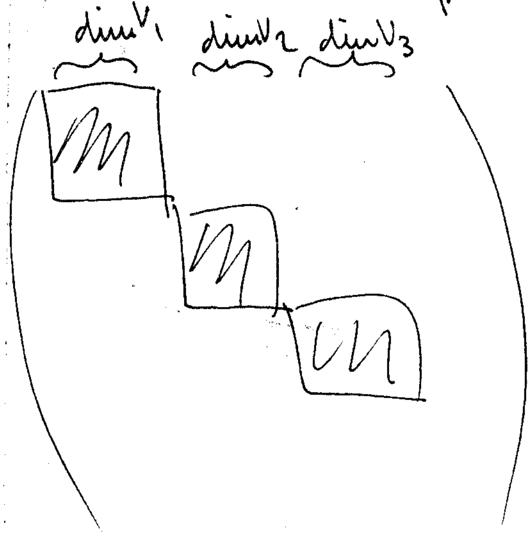
since any decomp into s_i's for w can be obtained from one decomp w/ these 3 transformation

$\mathbb{C}[G]$ elts of it : $\sum_{g \in G} \alpha_g \cdot g$ ← can think of (1) as a function on the group
long vector of α_g's.

(2) operators acting on $\oplus V_i$
↑
sum of all irreps

Theorem. $\mathbb{C}[G] \simeq \oplus \, \mathrm{End}_{\mathbb{C}}(V_i)$ ← isom. of algebras.
V_i irrep
not reps.
↑ group of all matrices from V_i to V_i

as reps. $\mathbb{C}[G] \simeq \oplus (\dim V_i)$



$\dim \mathbb{C}[G] = |G| = \sum \dim(V_i)^2$

Center of $\mathbb{C}[G]$ is $Z_{\mathbb{C}[G]} = \{ a \in \mathbb{C}[G] \mid a \cdot g = g \cdot a \quad \forall g \in$ ~~$\mathbb{C}[G]$~~ $\}$

$g \in G$
why not $\mathbb{C}[G]$
bec it is ⟺.

group elts as fncts on G:

$\mathbb{C}[G] \ni a = \sum \alpha_g \, g \qquad a \cdot h = h \cdot a$

Lemma. (1) $Z_{\mathbb{C}[G]} \simeq \mathbb{C}_{\text{class}}(G)$, basis $\langle C \rangle = \sum_{g \in C} g$ ← conj. class C

(2) $Z_{\mathbb{C}[G]} = \left\{ \begin{pmatrix} \boxed{aI} & & 0 \\ & \boxed{bI} & \\ 0 & & \boxed{cI} \ldots \end{pmatrix} \right\}$ diag matrices w/ same entries in each block.

Suppose have rep $V = V_1 \oplus V_2 \oplus \dots^{(*)}$ s.t. all multiplicities are 1

V has simple multiplicities

Lemma. Decomp (*) is unique. (not just up to isom, but just u

if have matrix w/ different eigenvalues → can uniquely pick eigenvectors.

irreps are generalized eigenvectors.
If we have 2 same eigenvalues → eigenspaces...

Proof. By Schur lemma $\dim \mathrm{Hom}_G(V_i, V) = 1 \Rightarrow$
V_i = Image of a G-homom $V_i \to V$
V_i can be chosen in a unique way.
nonzero

Okounkov - Vershik construction of reps of S_n

I Gelfand - Tsetlin basis $\{1\} = G_0 \subset G_1 \subset G_2 \subset \dots$ sequence of finite groups G_i

$G_n^\vee$ = set of (equivalence classes) of irreps of G_n

V_λ - irreducible G_n-module for $\lambda \in G_n^\vee$

Assume that $\mathrm{Res}^{G_n}_{G_{n-1}} V_\lambda$ has simple multiplicities.

Branching graph (aka Brattelli diagram) kind of generaliz. of Young lattice.

elts $G_0^\vee \cup G_1^\vee \cup .. \cup G_n^\vee \cup \dots$ edges $\mu \nearrow \lambda$ if V_μ appears in $\mathrm{Res}^{G_n}_{G_{n-1}} V_\lambda$. , $\lambda \in G_n^\vee$, $\mu \in G_{n-1}^\vee$.

by lemma ↓ this decomp is un

this decomp is unique ↙

$$V_\lambda = \bigoplus_{\mu \nearrow \lambda} V_\mu = \bigoplus_{\nu \nearrow \mu \nearrow \lambda} V_\nu = \dots = \bigoplus V_T$$

restrict to G_{n-2}

$T = \lambda^0 \nearrow \lambda^1 \nearrow \dots \nearrow \lambda$

↑ analogue of Young tableau: a path in Brattelli diagram.

V_T is 1-dim'l.

Pick vector v_T that generates V_T.

↑

! unique up to scalars since V_T unique !

The basis $\{ v_T \mid T = \lambda^0 \nearrow \lambda^1 \nearrow \dots \nearrow \lambda \}$ is called the Gelfand-Tsetlin basis.

Let $Z_n = Z_{\mathbb{C}[G_n]}$.

Def: Gelfand-Tsetlin subalgebra $GT_n \subset \mathbb{C}[G_n]$ is defined by $GT_n = \langle Z_1, Z_2, \dots, Z_n$

↑ Spanned by, generated by in $\mathbb{C}[G_n]$.

Since $G_1 \subset G_2 \subset \dots \Rightarrow \mathbb{C}[G_1] \subset \mathbb{C}[G_2] \subset \dots \Rightarrow Z_1, \dots, Z_n \in \mathbb{C}[G_n]$

Claim. GT_n is commutative.

a priori elts in Z_2 might not commute w/ elts in G_n.
but it's still easy!

pick elt in Z_n ⇒ commutes w/ all elts in $\mathbb{C}[G_n]$, in part.
— " " — $Z_1, \dots, Z_{n-1}$.

⇒ commutative.

Gelfand-Tsetlin

Pick the GT-basis in each V_λ.

$\mathbb{C}[G_n] = \bigoplus \operatorname{End} V_\lambda$ given by some particular matrices
algebra of block diagonal matrices

we want it in this nice basis, GT-basis

<u>Lemma.</u> (1) GT_n = algebra of all diagonal matrices (w.r.t. GT-basis)

(2). GT_n is maximal commutativ subalgebra of the group alg $\mathbb{C}[G_n]$.

For matrices: maximal comm. alg. $\boxed{\text{in matrices}}$ is the alg of all diagonal matrices.

Okounkov - Vershik construction (cont'd)

09/21/06

$G_0 \subset G_1 \subset G_2 \subset \dots$

V_λ irreps of G_n, $\lambda \in G_n^\vee$

Assume $\operatorname{Res}^{G_n}_{G_{n-1}} V_\lambda$ has simple multiplicities

Branching graph on $G_0^\vee \cup G_1^\vee \cup G_2^\vee \cup .$

GT-basis in V_λ labelled by paths in this graph. $\{ v_T \mid T = \lambda^0 \nearrow \lambda^1 \nearrow \dots \nearrow \lambda \}$

$Z_n = Z_{\mathbb{C}[G_n]}$

GT-algebra $GT_n = \langle Z_1, .., Z_n \rangle$ subalgebra of $\mathbb{C}[G_n]$.

$$\mathbb{C}[G_n] \simeq \bigoplus_{\lambda \in G_n^\vee} \operatorname{End}(V_\lambda) = \begin{pmatrix} \blacksquare & & 0 \\ & \blacksquare & \\ 0 & & \ddots \blacksquare \end{pmatrix}$$

Lemma. (0) GT_n is commutativ bec any elt in Z_i commutes w/ everything before (in $Z_1, , Z_{i-1}$)

(1) GT_n - algebra of diagonal matrices w.r.t. GT-basis in each V_λ

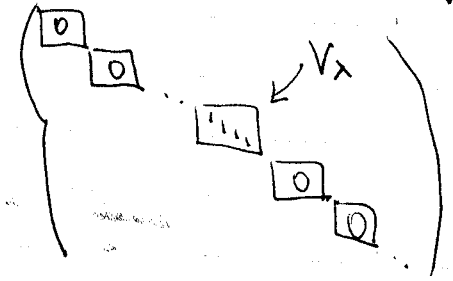
(2) GT_n - maximal commutative subalgebra in $\mathbb{C}[G_n]$.

(3) $v \in V_\lambda$ is in GT-basis iff v is a common eigenvector of elts of GT_n

(4) Each basis elt is uniquely determined by eigenvalues of elts of GT_n.

Only nontrivial statement is (1); (2), (3), (4) easily follow, and are abt matrices not of GT-bc

Proof of (1). $P_\lambda =$



$$P_\lambda := \bigoplus_{\mu \in G_n^\vee} V_\mu \longrightarrow V_\lambda$$

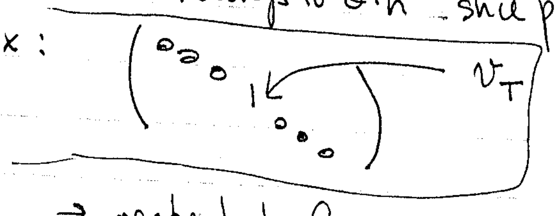
projection.

$P_\lambda \in Z_n$ since same entries in each block so it commutes w/ any other block diagonal matrix

$T = \lambda^0 \uparrow \lambda^1 \uparrow \cdots \uparrow \lambda$ path

construct operator for this path: $P_T = \underset{\overset{\Uparrow}{Z_0}}{P_{\lambda^0}} \underset{\overset{\Uparrow}{Z_1}}{P_{\lambda^1}} \cdots \underset{\overset{\Uparrow}{Z_{n-1}}}{P_{\lambda^{n-1}}} \underset{\overset{\Uparrow}{Z_n}}{P_\lambda} \in GT_n$

P_T corresponds to a matrix:



belongs to GT_n since product of elt of centers

$P_\lambda := \bigoplus_{\mu \in G_n^\vee} V_\mu \to V_\lambda$ → restrict to G_{n-1} → to G_{n-2} ··

So project into some piece ··

$P_T = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$

$\Rightarrow$ {all diagonal matrices} $\underline{\in}$ GT_n. Want to prove they are equal.

max comm. alg in ~~matrices~~ alg of matrices — this is comm.

$\Rightarrow$ {all diag matr.} $= GT_n$.

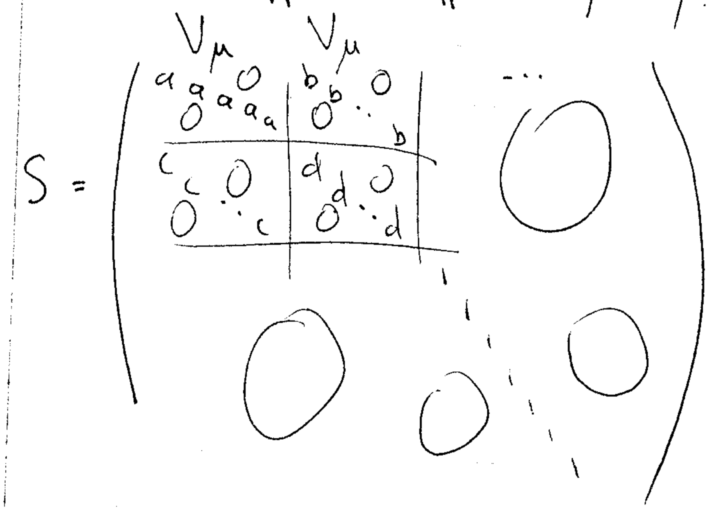
<u>From now on $G_n = S_n$.</u> $S_0 \subset S_1 \subset S_2 \subset ..$ assume that they are <u>embedded</u> in the <u>standard</u> way.

Need: $\operatorname{Res}^{S_n}_{S_{n-1}} V_\lambda$ has simple multiplicities

<u>Def.</u> $A \supset B$ subalgebra
centralizer $Z(A, B) = \{ a \in A \mid ab = ba \quad \forall b \in B \}$
if $B = A \to$ this is center, $A \neq B \to$ can be bigger.

<u>Lemma.</u> $H \subset G$ two finite groups. TFAE:
(1) Res^G_H have simple multiplicities
(2) $Z(\mathbb{C}[G], \mathbb{C}[H])$ is commutative

<u>Proof.</u> (2) $\Rightarrow$ (1) Suppose $\operatorname{Res}^G_H V_\lambda = V_\mu \oplus V_\mu \oplus ..$ now show $Z(\mathbb{C}[G], \mathbb{C}[H])$ is then not commutative



$\in \operatorname{End}(V_\lambda) \subset \mathbb{C}[G]$

sum of all $\operatorname{End}(V_\mu)$..

acts on V_λ

Claim that all these matrices commute w/ the action of H.

action of H:

can pick 2 bases of V_μ (maybe different)
s.t. the matrices are same: A.

$\Rightarrow S \subset Z(\mathbb{C}[G], \mathbb{C}[H])$ Contradiction as S is not commu

since $S \subseteq GL_2$; & GL_2 is not commutative

$R: G \to GL(V_\lambda)$

$S \cdot R(h) = R(h) \cdot S \quad \forall h \in H$

$\downarrow$

$R(h)$ has form $\begin{pmatrix} \boxed{A} & & \\ & \boxed{A} & \\ & & \boxed{C} \end{pmatrix}$

$V_\lambda = \underset{\uparrow}{V_\mu} \oplus \underset{\uparrow}{V_\mu} \oplus W$

picked bases
so that rep given by same matrices

$R(h) = \begin{pmatrix} \boxed{C} & & \\ & \boxed{C} & \\ & & \boxed{D} \end{pmatrix}$

Exercise. Prove that (1) $\Rightarrow$ (2).

Need: $Z_{n-1,1} = Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$ is commutative

→ want to fix n & could permute any others.
so in a perm we have a (cycle) partition w/ 1 marked part (which contains n) everything else we can again permute.

$Z_n = Z_{\mathbb{C}[S_n]}$ has basis ; $n = c_1 + c_2 + \dots + c_k$

$$[c_1, \dots, c_k] = \sum_{\substack{w \text{ has cycles} \\ \text{of sizes } c_1, c_2, \dots}} w \in \mathbb{C}[S_n]$$

Def. Marked partition $n = \bar{c}_1 + c_2 + + c_k$

$$[\bar{c}_1, c_2, \ldots, c_k] := \sum_{w \in S_n} w \quad \in \mathbb{C}[S_n].$$

with cycle sizes $c_1 \ldots c_k$
s.t. $n \in$ cycle of length c_1

Example. $[3, 2, \bar{2}] = \sum (i_1, i_2, i_3)(i_4, i_5)(i_6, 7)$
$\{i_1 .. i_6\} = \{1, .., 6\}$

Claim. These elts $\underset{[\bar{c}_1, .., c_k]}{}$ form a basis of $Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}]) = Z_{n-1,1}$.

Proof. $\sum \alpha_w \cdot w \in \mathbb{C}[S_n]$ belongs to $Z_{n-1,1}$ iff $\forall u \in S_{n-1} \subset S_n$ standard embeddin

$$\sum_w \alpha_w \cdot w = \sum_w \alpha_w u^{-1} w u = \sum_w \alpha_{u^{-1} w u} w$$

$$? \rightarrow \quad \alpha_w = \alpha_{u^{-1} w u} \quad \ldots \quad \blacksquare$$

Exercise. Prove that if $a = [\bar{c}_1, c_2, .., c_k]$, $b = [\bar{d}_1, d_2, .., d_l]$ then $ab = ba$
in a direct way.

Jucys - Murphy elements

$$X_i = (1, i) + (2, i) + (3, i) + \cdots + (i-1, i) \in \mathbb{C}[S_n]$$

(1) $X_i \in GT_n$, the GT-algebra

$$X_i = [\bar{2}, \overbrace{1, 1 .. 1}^{n-2}] \in \mathbb{C}[S_i] \subset \mathbb{C}[S_n]$$

using $[\bar{c}_1, c_1, ..]$ notation.

sum of all transp. of $1..i$

$$X_i = [2, \underbrace{1 .. 1}_{\substack{i-2 \ \cap \\ Z_i}}] - [2, \underbrace{1, .., 1}_{\substack{i-3 \ \cap \\ Z_{i-1}}}] \longrightarrow = (1, i) + \ + (i-1, i)$$

$\sum$ all transp of $1 .. i-1$

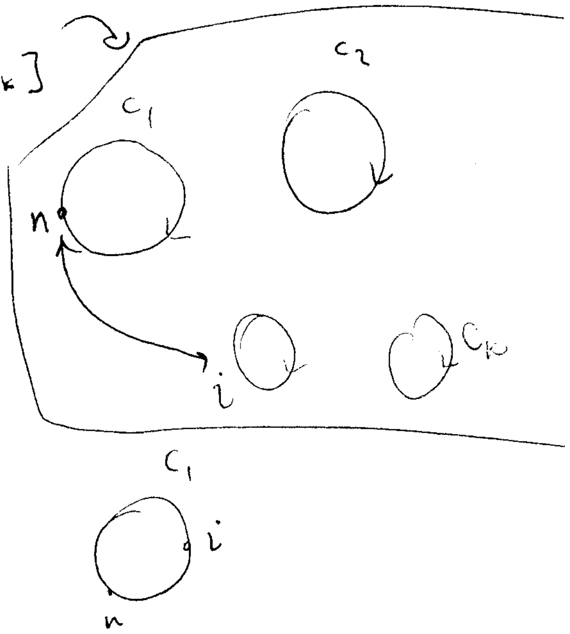
Lemma./ Proposition. $Z_{n-1,1} = \langle Z_{n-1}, X_n \rangle$

Proof. It is clear that $Z_{n-1}, X_n \in Z_{n-1,1}$. $\Rightarrow$ $\langle Z_{n-1}, X_n \rangle \subset Z_{n-1,1}$

$$X_1 = (1,n) + (2,n) + \dots + (n-1,n) = \sum_{i=1}^{n-1} (i,n)$$

Need to check that any $[\bar{c}_1, .., a_k]$ w/ $\sum_{i=1}^{k} c_i = n$ can be expressed in terms of X_n and $\underset{[\bar{1}, a_1, .., a_l] \in Z_{n-1,1}}{\underset{\parallel}{[a_1, .., a_l] \in Z_{n-1}}}$

$$X_n \cdot [\bar{c}_1, c_2, .., c_k] = \underbrace{\sum_{c_1 = c_1' + c_1''} \alpha_u [\bar{c}_1', c_1'', c_2 .. c_k] +}\sum_{i=2}^{k} \gamma_i [\overline{c_1 + c_i}, c_2, .., \hat{c}_i, .. c_k]$$



Need to express $[\bar{a}_1, a_2, .., a_l]$

Assume by induction that we already expressed all $[\bar{c}_1, .., c_k]$ s.t. $k > l$ or $(k = l$ & $c_1 > a_1)$

Base of induction. $[\bar{1}, 1, 1, .. 1] = \overset{\text{identity perm}}{1} = X_n^0$.

If $a_1 = 1$ then done, because $[\bar{1}, a_2, .. a_l] = [a_2, .., a_l] \in Z_{n-1}$. ✓
Otherwise: $X_n \cdot [\overline{a_1 - 1}, 1, a_2, .. a_l] = \underset{\text{split } a_1 - 1}{\sum \alpha [\bar{\ } \dots]} + [\bar{a}_1, a_2, .. a_l] + \sum_{i=2}^{l} \beta_i [\overline{a_1 a_i}, \dots]$ merge $a_1 - 1 + a_i$

anything expressed by ind except

$\Rightarrow$ so we get an expression for $[\bar{a}_1, a_2, .. a_l]$ it as desired

bigger # of parts so done by ind.

We can express $[\bar{a}_1, a_2, .. a_\ell]$ in terms of stuff that we have already expressed in terms of X_n as needed.

Exercise. ... bit later.

Corollary: $Z_{n-1,1}$ is commutative $\Rightarrow$ $\mathrm{Res}^{S_n}_{S_{n-1}}$ have simple multiplicities.

Proof: Z_{n-1} and X_n commute w/ each other, because X_n clearly belongs to centralizer... think abt this.

forget abt last letter → symm w.r.t. first n-1 letters

Corollary. $GT_n = \langle X_1, X_2, .., X_n \rangle$ algebra of all expressions in X_i!

Proof. Assume by induction $GT_{n-1} = \langle X_1, .., X_{n-1} \rangle$

Clearly $GT_n \geq \langle GT_{n-1}, X_n \rangle$. Need $\leq$. Need $Z_n \subset \langle GT_{n-1}, X_n \rangle$.

$$Z_n < Z_{n-1,1} \underset{\substack{\uparrow \\ \text{by Prop.}}}{=} \langle Z_{n-1}, X_n \rangle \subset \langle GT_{n-1}, X_n \rangle. \quad ▨$$

09/26/06

Last time: • each irrep V_λ has "nice" GT-basis $\{v_T\}$ $\quad T = \lambda^0 \nearrow .. \nearrow \lambda$

• JM-elements $X_k = \sum_{i=1}^{k-1} (i,k) \in \mathbb{C}[S_n]$, $k = 1, .., n$

generate GT-algebra, a maximal commutative subalgebra of $\mathbb{C}[S_n]$

• GT-basis is the unique basis s.t. basis elts are common eigenvectors of the X

$X_i \cdot v_T = a_{i,T} v_T \qquad \alpha(T) = (a_{1,T}, .., a_{n,T}) \in \mathbb{C}^n$

↳ uniquely characterize basis elts.

↳ this is only property we'll use today.

all elts in GT-basis

$\mathrm{Spec}(n) = \{ \alpha(T) : \text{for all paths } T \}$

For $\alpha, \beta \in \mathrm{Spec}(n)$, $\alpha \sim \beta$ if they correspond to same repre

$\hookrightarrow$ if v_α and v_β are in the same irrep V_λ

$v_\alpha = v_{\alpha(T)} = v_T$

We need to describe Spec(n), and the equivalence classes of $\sim$ since then we have all irreps.

<u>Example.</u> S_n

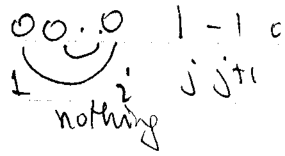
standard $(n-1)$-dim rep, lives in space $V = \{ (x_1, .., x_n) \mid x_1 + \ldots + x_n = 0 \} \subseteq \mathbb{C}$

Pick $\underbrace{\text{some basis}}_{\downarrow \text{ not GT-basis.}}$ of V: $v_j = e_j - e_{j+1} = (0 .. 0 \ \underset{\overset{\uparrow}{j}}{1} \ \underset{\overset{\uparrow}{j+1}}{-1} \ 0 .. 0)$

$X_i = (1,i) + \ldots (i-1, i)$

$X_i v_j = ?$

I) $j > i$, $X_i v_j = (i-1) v_j$



II) $i \geq j+2$, $X_i v_j = (i-3) v_j + \cancel{e_i} - \underbrace{e_{j+1} + \cancel{e_j}}_{v_j} - \cancel{e_i}$

$X_i v_j = (i-2) v_j$

III) $i = j$ $\quad X_i v_j = v_1 + \ldots + v_j +$
$v_2 + \ldots + v_j +$
$v_3 + \ldots v_j +$
$\ddots \quad =$

$1000 .. \overset{j+1}{-1}00 = v_1 + v_2 + \ldots + v_j$

$= v_1 + 2 v_2 + 3 v_3 + \ldots + (j-1) v_{j-1} + (j-1) v_j$

IV $j = i+1$ $\quad X_i v_j = -v_1 - 2v_2 - 3v_3 - \ldots - (i-2) v_{i-2} - 1 v_{i-1}$

Operator given by matrix

(n-1) x (n-1) matrix

$$X_i = \left(\begin{array}{cccc|cc|ccc} \multicolumn{4}{c|}{\overbrace{\qquad\qquad\qquad}^{i-2}} & & & & & \\ i-2 & & & 0 & -1 & 1 & & & \\ & i-2 & & & -2 & 2 & & 0 & \\ & & \ddots & & & & & & \\ 0 & & & i-2 & -(i-2) & i-2 & & & \\ & & & & -1 & i-1 & & & \\ & & & & & i-1 & & & \\ \hline & & & & & & i-1 & & 0 \\ & 0 & & & & & & \ddots & \\ & & & & & & 0 & & i-1 \end{array}\right)$$

$i \geq 2$

$X_1 = 0$

by claim - exists unique basis where we can diagonalize matrices X_i at same time. Find this basis, which is in fact the GT-basis.

Let GT-basis be $\tilde{v}_i$. In GT-basis these operators X_i are given by

$$X_i = \begin{pmatrix} \overbrace{\begin{matrix} i-2 & & 0 \\ & \ddots & \\ & & i-2 \end{matrix}}^{i-2} & & & & 0 \\ & -1 & & & \\ & & i-1 & & \\ 0 & & & i-1 & \\ & & & & \ddots \; i-1 \end{pmatrix}$$

just the diagonal from X_i above

Let $n=4$ $\quad X_1 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}$ $\quad X_2 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ $\quad X_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 2 \end{pmatrix}$

$X_4 = \begin{pmatrix} 2 & & \\ & 2 & \\ & & -1 \end{pmatrix}$

↑ JM elts look like this in the GT-basis.

in Spec(n) ← $\alpha(\tilde{v}_1) = (0,-1,1,2)$, $\alpha(\tilde{v}_2) = (0,1,-1,2)$ $\quad \alpha(\tilde{v}_3) = (0,1,2,-1)$

↑ first basis elt; just look at first entries of operators of X_i

$\alpha(\tilde{v}_1), \alpha(\tilde{v}_2), \alpha(\tilde{v}_3)$

all start w/ 0, entries integers

$\alpha(\tilde{v}_1) = (a_1 \dots\ )$ eigenvalues → read off from diagonal.

$\alpha(\tilde{v}_i)$ - describes one conj. class.
$\alpha(\tilde{v}_i)$'s are in Spec (n).

vectors $\alpha(\tilde{v}_i)$ have entries → permutation of each other

Recall $\mathbb{C}[S_n]$ is generated as an algebra by $s_1, .., s_{n-1}$ (adj transp) with relations $s_i^2 = 1$, $s_i s_j = s_j s_i$ $\quad |i-j| \geqslant 2$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

Relations as for S_n - difference here we not only can multiply elts, but also add them since we have an algebra.

<u>Theorem.</u> JM-elements

1) $X_i X_j = X_j X_i \quad \forall i, j$
2) $s_i X_j = X_j s_i$ unless $j = i, i+1$
3) $s_i X_i + 1 = X_{i+1} s_i$

(*)

algebra with these relations is called the degenerate Hecke algebra DHA

↘ gen. by X_i's and s_i's.

<u>Proof.</u> → just check the relations. Let's just do 3)

$$s_i X_i + 1 = (i, i+1)\big((1,i) + (2,i) \cdots + (i-1,i)\big) + 1 =$$

$$= \big((1,i+1) + \ + (i,i+1)\big)(i\ i+1) = X_{i+1} s_i .$$

Local analysis of Spec(n)

Pick vector $v \in$ GT-basis. → $\alpha(v) = (a_1, .., a_n)$, let $a_i = a$, $a_{i+1} = b$

so $X_i v = a \cdot v$, $X_{i+1} v = b v$

want to figure out what can we say abt numbers a and b?

$w = s_i v$

I v & w are lin. dependent $\Rightarrow$ $w = \pm v$ (*) $\Rightarrow$ $b = a \pm 1$

($w = s_i v$

$s_i w = s_i s_i v = \underline{V}$. → so coefficient coordinate should be ± 1 differ by)

$(s_i X_i + 1) v = X_{i+1} s_i v$

$\pm a v + v = b(\pm v)$

II v & w lin indep. $\Rightarrow$ v & w generate 2-dim subspace

On this subspace X_i, X_{i+1}, s_i act by the matrices

(*) $\Rightarrow$ $X_i = \begin{pmatrix} a & -1 \\ 0 & b \end{pmatrix}$ $X_{i+1} = \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix}$ $s_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Claim $a \neq b$

if $a = b$ then $X_i = \begin{pmatrix} a & -1 \\ & a \end{pmatrix}$

has a nontrivial Jordan block, $\Rightarrow$ X_i not diagon

→ but we know in GT-basis they can be diagonalized! ↯

Let $\tilde{v} = v + (b-a) w$

Instead of using v, w → use basis $v, \tilde{v}$

$X_i : \tilde{v} \longmapsto b \tilde{v}$ (check by matrices relations above) since we already know how X_i acts on v, w

$X_{i+1} : \tilde{v} \longmapsto a \tilde{v}$

$s_i \tilde{v} = w + (b-a) v$, since s_i switch w & v. $\Rightarrow$ $\hat{v} \in$ GT-basi

$$\text{If } \underbrace{b = a \pm 1}_{} \Rightarrow s_i \overset{\in \text{GT-basis}}{\boxed{\tilde{v}}} = \pm \tilde{v} \overset{\text{by I since } \tilde{v},\, s_i\tilde{v} \text{ lin dep.}}{\Rightarrow} \underbrace{a = b \pm 1}_{} \quad ↯$$

$\Rightarrow b \neq a \pm 1$

$\tilde{v} \in$ GT-basis.

What we just proved is:

Proposition. Let $\alpha = (a_1, .., a_n) \in \mathrm{Spec}(n)$. Then

1) $a_i \neq a_{i+1} \quad \forall i$

2) if $a_{i+1} = a_i \pm 1$ then $s_i v_\alpha = \pm v_\alpha$

3) if $a_i \neq a_{i+1} \pm 1$ then $\alpha' = (a_1, .., a_{i-1}\ a_{i+1}\ a_i \cdots a_n) \in \mathrm{Spec}($

$\alpha' \sim \alpha$.

We'll call transposition $(a_1, .., a_n) \to (a_1 .. a_{i+1}\ a_i \cdots a_n)$ if $a_{i+1} - a_i \neq \pm 1$

admissible transposition. By 3) if $(a_1 .., a_n) \in \mathrm{Spec}(n) \Rightarrow (a_1 .. a_{i+1}\ a_i .. a_n) \in \mathrm{Sp}$

Def. ~~Cont(n) ⊆ $\mathbb{Z}^n$ and $(a_1, ..., a_n)$~~

Cont(n) is the set of vectors $(a_1, .. a_n)$ (~~in $\mathbb{Z}$~~)^n s.t.

1) $a_1 = 0$

2) $\forall i \quad a_i - 1$ or $a_i + 1 \in \{a_1 \dots a_{i-1}\}$

3) If $a_i = a_j = a$ for $i < j$ then $\{a_{i+1}, ..., a_{j-1}\} \ni a-1, a+1$

} ⇒ they are integers already implied.

Eventually we'll prove Cont(n) = Spec(n)

Claim

$\mathrm{Spec}(n) \subseteq \mathrm{Cont}(n)$

Theorem. $\mathrm{Spec}(n) \subseteq \mathrm{Cont}(n)$

Assume $(a_1, .., a_n) \in \mathrm{Spec}(n)$

Proof. 1) $X_1 = 0$ → can have only 0 eigenvalues → $a_1 = 0$

2) $X_2 = s_1$ → only possible eigenvalues ±1 → $a_2 = \pm 1$

Suppose that condition 2) fails for i

Using admissible transpositions we can transform $(a_1, \dots, a_n) \rightsquigarrow (\underset{\overset{\|}{0}}{a_1}, \underset{\uparrow}{a_i}, \dots)$

can only have ±1 on 2nd po

$a_i = \pm 1$, adjacent # before it. Contradiction.
so cannot switch 0, 1, or 0, -1

could also make argument of moving
a_i to 1st position → but then 0 →
and again contradiction?

3) Suppose 3) fails.

Find "bad" interval $(a_i .. a_j)$ where $a_i = a_j$, of minimal possible length in

Spec(n). We cannot have • (a, a)

• $(a, b, \dots, a)$ where $b \neq a \pm 1$

• $(a, \underset{\overset{\uparrow}{\text{cannot be an } a \text{ inside.}}}{a+1, \dots,} a+1, a)$ bec then would be then shorter "bad" interval

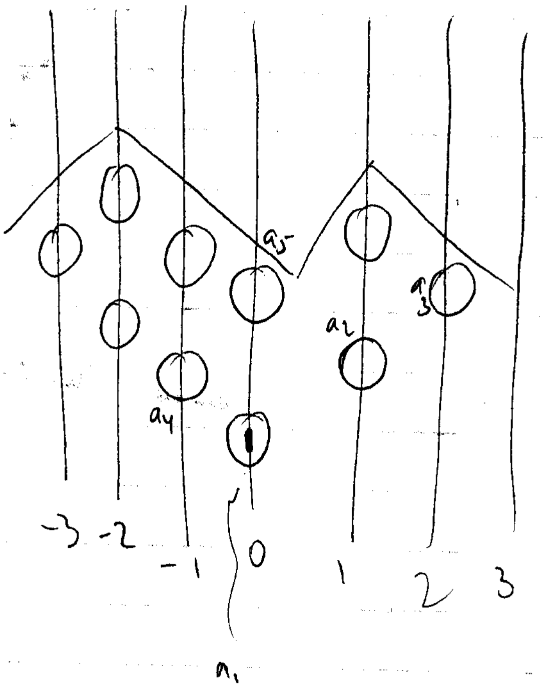
• $(a, a+1, .., a-1, a)$ – bec it is not bad.
• $(a \; a+1 \; a)$

$\alpha = (.. \overset{i}{a}\ \overset{i+1}{a+1}\ \overset{i+2}{a} ..)$. By prop 2)

$$\overbrace{s_i \overbrace{s_{i+1} \overbrace{s_i v_\alpha}^{v_\alpha}}^{-v_\alpha}}^{-v_\alpha} = -v_\alpha$$

$$\overbrace{s_{i+1} \overbrace{s_i \overbrace{s_{i+1} v_\alpha}^{-v_\alpha}}^{-v_\alpha}}^{v_\alpha} = v_\alpha$$

so $v_\alpha = -v_\alpha$ ↯ since it is a basis vector.



follows that a_i's are integers.