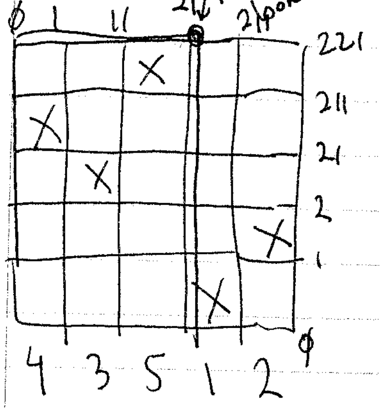
10/12/06



for partition here look at 5x3 rectangle.

Lemma. The collection of partitions $\lambda(w(i,j))$ satisfy the local rules for growth diagrams.

↳ $P = \begin{array}{|cc|} 1 & 2 \\ 3 & 5 \\ 4 & \end{array}$ $Q = \begin{array}{|cc|} 1 & 3 \\ 2 & 5 \\ 4 & \end{array}$

Theorem. The map $w \mapsto (P, Q)$ is the Schensted correspondence.

Corollary. $w \xmapsto{\text{Schensted}} (P, Q)$ of shape λ. Then $\lambda_1 =$ ~~maximal~~ possible size of increasing subword in w.

~~Example~~ **Def.** w is 123-avoiding if $\neg\exists\ i<j<k$ s.t. $w(i)<w(j)<w(k)$.

Corollary. $\left\{\begin{array}{c}\text{123-avoiding} \\ \text{permutations}\end{array}\right\} \xrightarrow{\text{Schensted}} (P, Q)$ of shape λ s.t. $\lambda_1 \leq 2$

↓

longest increasing subsequence is of length 1 or 2, $\Rightarrow \lambda_1 \leq 2$.

↳ solves problem 1 on PSET.

Dual RSK.

$A = (a_{ij})$ $\infty \times \infty$ matrix with finite support s.t. $a_{ij} \in \{0,1\}$

$A \xleftrightarrow{RSK^*} (P, Q)$ P, Q SSYT's w/ weight(P) = row sums of A

weight(Q) = column sums of

shape$(P) = \lambda$, ~~#~~ shape$(Q) = \lambda'$ ← conjugate parti

Dual Cauchy Identity.

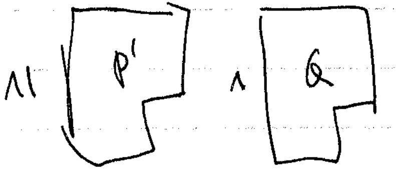
$$\sum_{\lambda} s_\lambda(x_1, x_2, \ldots) s_{\lambda'}(y_1, y_2, \ldots) = \prod_{i,j} (1 + x_i y_j)$$

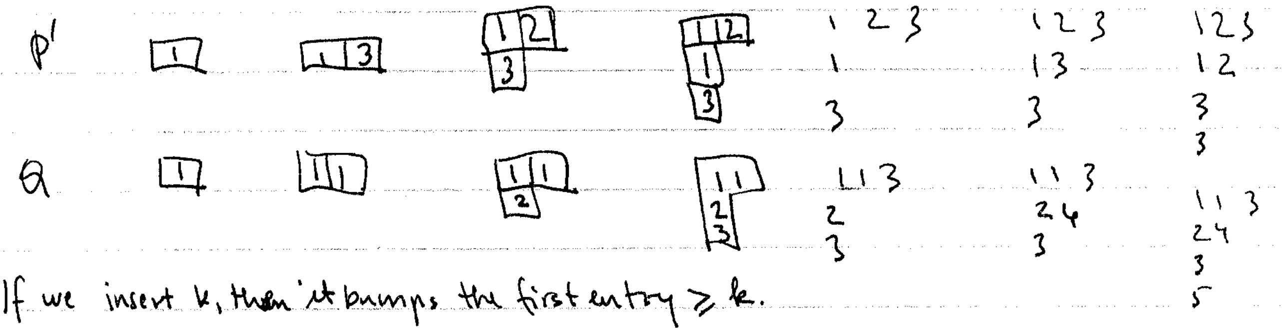
$$A \rightsquigarrow w_A$$

$$w_A = \begin{pmatrix} 1 & 1 & 2 & 3 & 3 & 4 & 5 \\ 1 & 3 & 2 & 1 & 3 & 3 & 2 \end{pmatrix}$$ not allowed repeated columns; except this it's same as generalized permutation

$$\downarrow$$

(P', Q)





If we insert k, then it bumps the first entry $\geq k$.

Gelfand-Tsetlin patterns

$$\begin{array}{ccccccc} \lambda_1 & & \lambda_2 & & \lambda_3 & \cdots & \lambda_n \\ & \geq \mu_1 \geq & & \geq \mu_2 \geq & \cdots & \geq \mu_{n-1} \geq & \\ & & \geq \nu_1 \geq & \geq \nu_2 \geq & \cdots & \geq \nu_{n-2} \geq & \\ & & & \ddots & & & \\ & & \geq \delta_1 \geq & & \geq \delta_2 \geq & & \\ & & & \geq \gamma_1 \geq & & & \end{array}$$

Claim. GT-patterns w/ top row $\lambda = (\lambda_1, \lambda_2 \ldots \lambda_n)$ are in bijection with SSYT's of shape λ filled with entries $\leq n$.

~~1111 222 35~~

22333566

$$\begin{array}{llllllll} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 5 \\ 2 & 2 & 3 & 3 & 3 & 5 & 6 & 6 \\ 3 & 4 & 4 & 5 & 6 \\ 5 & 5 & 6 \\ 6 \end{array} \longrightarrow \begin{array}{ccccccccccc} 8 & & 8 & & 5 & & 3 & & 1 & & 0 \\ & 8 & & 6 & & 4 & & 2 & & 0 & \\ & & 7 & & 5 & & 3 & & 0 & & \\ & & & 7 & & 5 & & 1 & & & \\ & & & & 6 & & 2 & & & & \\ & & & & & 3 & & & & & \end{array}$$

$\lambda^{(i)}$ is the partition filled w/ entries $\leq i$

$\leq \lambda^{(1)} \leq \dots \leq \lambda^{(n)}$

write parts of $\lambda^{(i)}$ in ith row from the bottom

$\lambda^{(i)}/\lambda^{(i-1)}$ is a horizontal strip for any i.

Claim. λ/μ is a horizontal strip if and only if $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots$

Example when GT-patterns are more convenient than ~~tableaux~~.

$\tilde{s}_i : T \leftrightarrow \tilde{T}$

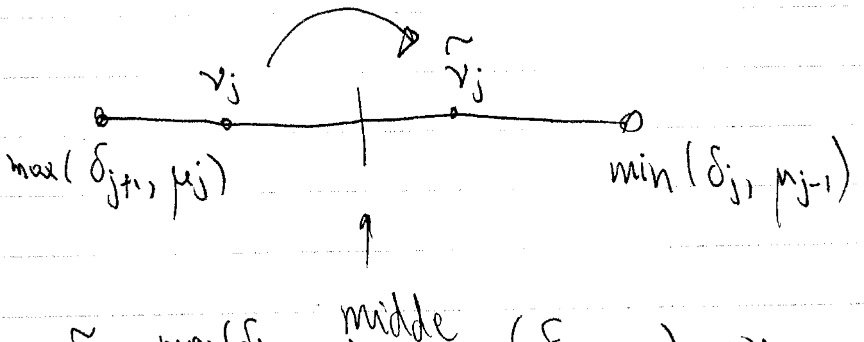
switches β_i w/ β_{i+1} in the weight of T

Operation $\tilde{s}_i$ on GT-patterns: only acts on ith row from bottom.

$$\left\backslash \begin{array}{ccc} \delta_1 & & \delta_{i+1} \\ & \nu_1, \dots \nu_i & \\ & \mu_1 \dots \mu_{i-1} & \end{array} \right/ \text{ i-th row}$$

$$\begin{array}{ccc} \delta_j & & \delta_{j+1} \\ & \nu_j & \\ \geq & & \geq \\ \mu_{j-1} & & \mu_j \end{array}$$

$\nu_j \in [\max(\delta_{j+1}, \mu_j), \min(\delta_j, \mu_{j-1})]$

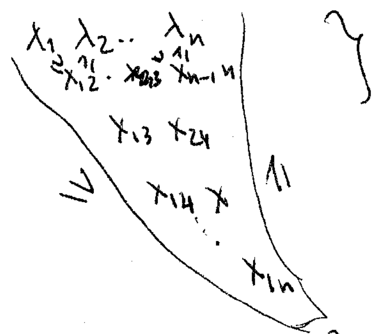


$\tilde{\nu}_j = \max(\delta_{j+1}, \mu_j) + \min(\delta_j, \mu_{j-1}) - \nu_j$

Claim. The operation $\tilde{s}_j : P \to \tilde{P}$ corresponds ~~to the transformation~~ replacing $(\nu_1 \dots \nu_i)$ w $(\tilde{\nu}_1 \dots \tilde{\nu}_i)$ in the corresponding GT-patterns.

GT-polytope : fix top row of GT-pattern

$GT(\lambda) = \{ (x_{ij}) \in \mathbb{R}^{\binom{n}{2}} \mid$



$\mathrm{Vol}(GT(\lambda)) \approx \#\{SSYT$ of shape λ with entries $\leq n\}$

$$\mathrm{Vol}(GT(n)) = \frac{1}{1!\,2!\cdots(n-1)!} \prod_{i<j} (\lambda_i - \lambda_j)$$

<u>Digression</u> $[n]_q = 1+q+q^2+\cdots+q^{n-1} = \frac{1-q^n}{1-q}$

$$[n!]_q = [1]_q \cdots [n]_q$$

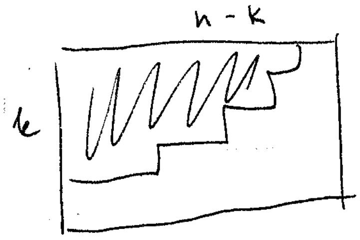
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]!\,[n-k]!}$$

<u>Theorem.</u> (1) $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial with positive integer coeffs.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = a_0 + a_1 q + a_2 q^2 + \cdots + a_N q^N \quad , \quad N = k(n-k)$$

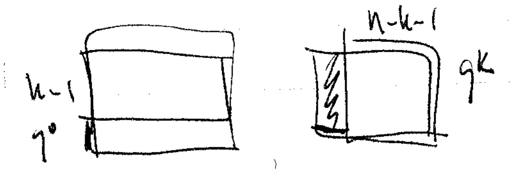
a_i are called Gaussian coeffs

(2) $a_i = \#$partitions λ s.t. $|\lambda| = i$ and $\lambda \leq k \times (n-k)$



<u>1st proof.</u> $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}.$

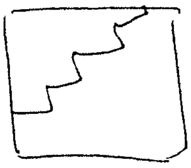
Let $\left\{ {n \atop k} \right\} = \sum_{\lambda \subset k \times (n-k)} q^{|\lambda|}$: $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + q^k$



10/17/06

Theorem. $\begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]_q!}{[k]_q!\,[n-k]_q!}$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}$$



2nd proof. $Gr_{kn}(\mathbb{F}_q)$ = set of k-dimensional linear subspaces in $(\mathbb{F}_q)^n$

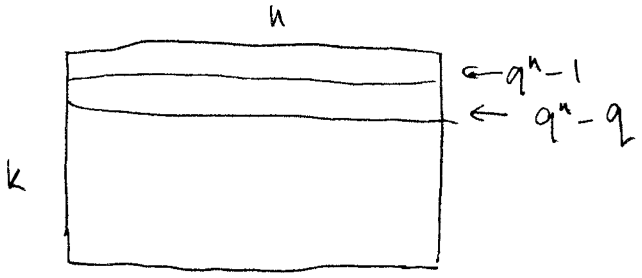
$q = p^k$, p-prime $\mathbb{F}_q$ - finite field w/ q elts

in more elementary terms:

$Gr_{kn}(\mathbb{F}_q)$ = set of $\{k \times n$ matrices w/ elts in $\mathbb{F}_q$ of rank $k\}$ / row operations

↑ Grassmannian

↑ modulo



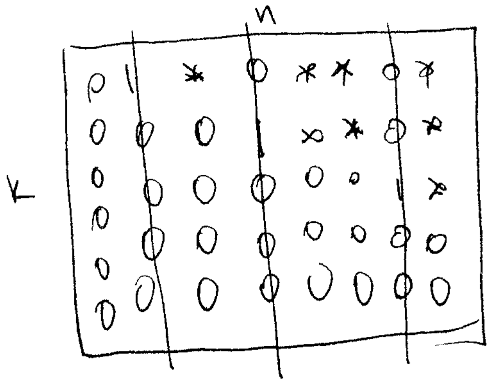
$$\# Gr_{kn}(\mathbb{F}_q) = \frac{(q^n-1)\cdot(q^n-q)(q^n-q^2)\cdots(q^n-q^{k-1})}{(q^k-1)\ (q^k-q)\cdots\ (q^k-q^{k-1})}$$

↙ row operations i.e. left action of nondeg. $k \times k$ matrices

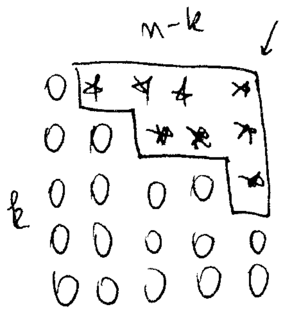
$$= \begin{bmatrix} n \\ k \end{bmatrix}_q$$

every matrix can be transformed by row operations into echelon form

Gaussian elimination



remove all ~~rows~~ columns that have 1 →



Young diagram inside $k \times (n-k)$ rectangle.

$\rightsquigarrow q^{\#\ *'s}$

$\Rightarrow$ Proved theorem for $q = p^k$

→ follows for all q

$$\begin{bmatrix}k+l\\k\end{bmatrix}_q = a_0 + a_1 q + \dots + a_{k\cdot l}\, q^{k\cdot l}$$

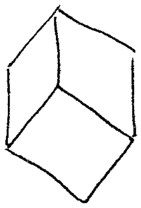
Theorem. The sequence $a_0, a_1, a_2 \dots$ is symmetric $(a_i = a_{kl-i})$ and unimodal.

$$a_0 \le a_1 \le \dots \le a_{\lfloor \frac{kl}{2} \rfloor} \ge \dots \ge a_{kl}$$

Sylvester's proof... but before: more general setting

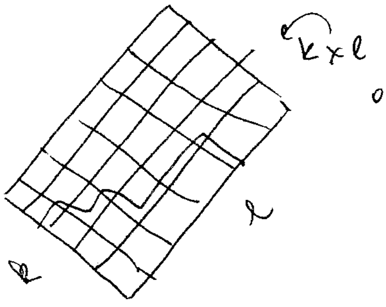
P poset (partially ordered set)

$\lambda \subseteq P$ is order ideal if $x \in \lambda, y < x \Rightarrow y \in \lambda$



Hasse diagram

$J(P)$ - lattice of order ideals



order ideals of poset in 1-1 corr. w/ Young diags that fit in k×l rectangle

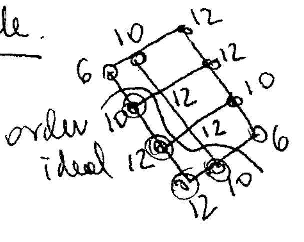
$J(P)_r$ = set of order ideals with r elements

For $\lambda \in J(P)$, let $\text{add}(\lambda) = \{ x \in P \setminus \lambda \mid \lambda \cup \{x\} \in J(P) \}$

$\text{remove}(\lambda) = \{ y \in \lambda \mid \lambda - \{y\} \in J(P) \}$

Lemma. Fix r. Suppose that there exists a weight function on P, $w : P \to \mathbb{R}_{>0}$ s.t. $\forall \lambda \in J(P)_r$ we have $\sum_{x \in \text{add}(\lambda)} w(x) > \sum_{y \in \text{remove}(\lambda)} w(y)$. Then $\# J(P)_r \le \# J(P)_{r+1}$

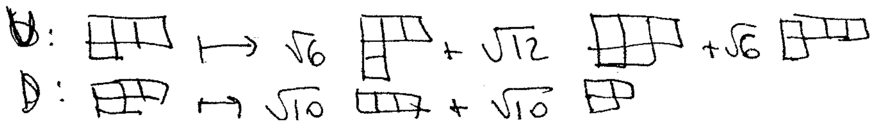
Example.



12	12	10	6
10	12	12	10
6	10	12	12

$\lambda =$ 

3 addible boxes: $6 + 12 + 6 > \underbrace{10 + 10}_{\text{removable boxes}}$



$\lambda \in J(P)$

<u>Proof.</u> U, D act on linear combinations of ~~Young diagrams~~ ~~λ that $\lambda \subseteq k \times \ell$~~

$$U: \lambda \to \sum_{\substack{\cancel{\mu = \lambda \cup \{x\}} \\ x \in \mathrm{add}(\lambda)}} \sqrt{w(x)}\, (\lambda \cup \{x\})$$

$$D: \lambda \longrightarrow \sum_{y \in \mathrm{remove}(\lambda)} \sqrt{w(y)}\, (\lambda \setminus \{y\})$$

<u>Claim</u> $H = DU - UD$ has diagonal form

$$H: \lambda \longmapsto \left(\sum_{x \in \mathrm{add}(\lambda)} \sqrt{w(x)}\sqrt{w(x)} - \sum_{y \in \mathrm{remove}(y)} \sqrt{w(y)}\sqrt{w(y)} \right) \lambda$$

$$DU = UD + H = \underset{\downarrow}{D^T \cdot D} + H$$

(symm, real, eigenvalues are ≥ 0

→ Hermitian matrix, positive semidefinite (all eigenvalues ≥ 0))

H is Hermitian positive definite (all eigenvalues > 0)

<u>Fact.</u> (positive semidefinite) + (positive definite) is a positive definite matrix.

Let V_r be space of linear combinations of $\lambda \in J(P)_r$

$$\begin{tikzcd}
 & V_{r+1} \arrow[dr, "D"] & \\
V_r \arrow[ur, "U"] \arrow[rr, "H"] \arrow[dr, "D"'] & & V_r \\
 & V_{r-1} \arrow[ur, "U"'] &
\end{tikzcd}$$

$DU: V_r \longrightarrow V_r$ is positive definite $\Rightarrow$ nondegenerate

DU has rank $= \dim V_r$

$DU: V_r \to V_{r+1} \longrightarrow V_r$ since rank $DU > \dim V_r \Rightarrow \dim V_{r+1} \geq \dim V_r$

$U = D^T$ because:

$U: v_i \mapsto \sum_j a_{ij} v_j$

in our case $a_{ij} = \sqrt{w(..)}$

$D: v_j \mapsto \sum_i a_{ij} v_i$

Take $w(x) = (l - c(x))(k + c(x))$

l

k

4·3	3·4	2·3	1·6
5·2	4·3	3·4	2·5
6·1	5·2	4·3	3·4

l

k

4·3	3·4	2·3	1·6
5·2	4·3	3·4	2·5
6·1	5·2	4·3	3·4

Lemma. $\sum_{x \in add(\lambda)} w(x) - \sum_{y \in remove(\lambda)} w(y) = k \cdot l - 2|\lambda|$ $(\star)$

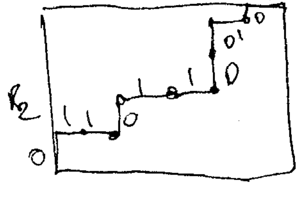
By above lemma we want $\# J(P)_r \leq \# J(P)_{r+1}$, so need $\sum_{x \in add} w(x) > \sum_{y \in remo} w(y)$

$\Rightarrow$ This works for $r = |\lambda| < kl/2$ we get: $a_0 \leq a_1 \leq \ldots \leq a_{\lfloor \frac{kl}{2} \rfloor}$.

$a_{\lfloor \frac{kl}{2} \rfloor} \geq \cdots \geq a_{kl}$ by symmetry.

So it remains to prove identity $(\star)$.

<u>Proof of lemma.</u> $W_\lambda = \sum_{x \in add(\lambda)} w(x) - \sum_{y \in remove(\lambda)} w(y)$



get sequence: $\underline{0\,1}\overset{\frown}{1}\,\underline{0}\overset{\frown}{1}\,\underline{0\,1}\overset{\frown}{1}\,0\,\underline{0\,1}\overset{\frown}{1}\,0\,0 = (\varepsilon_1 \ldots \varepsilon_{k+l})$ (not this picture though!)

addible boxes: where 0 followed by 1 !

removable boxes: 1 followed by 0.

$$W_\lambda = \sum_{i=1}^{k+l-1} (\varepsilon_{i+1} - \varepsilon_i) \cdot \underbrace{i \cdot (k+l-i)}_{\text{~~content~~ weight}} = (\varepsilon_2 - \varepsilon_1)(k+l-1) \cdot 1 + (\varepsilon_3 - \varepsilon_2)(k+l-2) \cdot 2 + \ldots$$

$$= (1-k-l) \cdot \varepsilon_1 + (3-k-l)\varepsilon_2 + (5-k-l)\varepsilon_3 + \ldots = -l(1+k+l) + 2 \sum_i i \varepsilon_i$$

Proof by induction. $\lambda = \emptyset$, $(\varepsilon_1 \dots \varepsilon_{k+l}) = (\underbrace{0 \, .. \, 0}_{k} \ \underbrace{1 \, .. \, 1}_{l})$ We have $W_\emptyset = k \cdot l$

Suppose $\mu = \lambda \cup \{x\}$

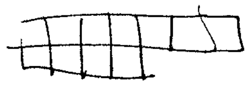
$\lambda = (\dots 01 \dots)$
$\mu = (\dots 10 \dots)$

$W_\mu = W_\lambda - 2$ and RHS $k \cdot l - 2|\lambda|$ also decreases by 2. $\boxtimes$

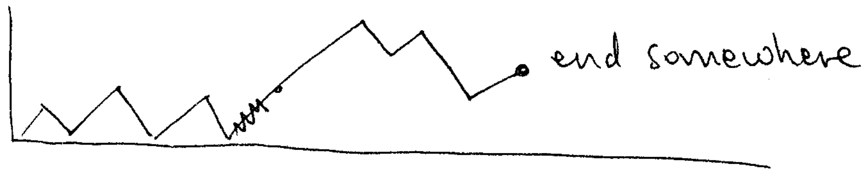
Action of SL_2 : $\dfrac{U, D, H}{E, F, H}$ → generate action of Lie algebra sl_2!

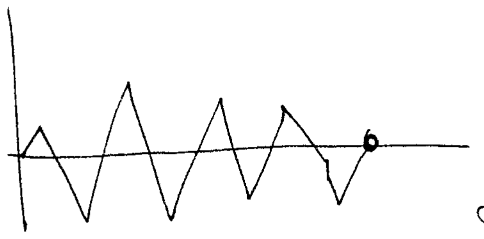
usually, this kind of unimodality results come from representation theory → actions of.

1 b)

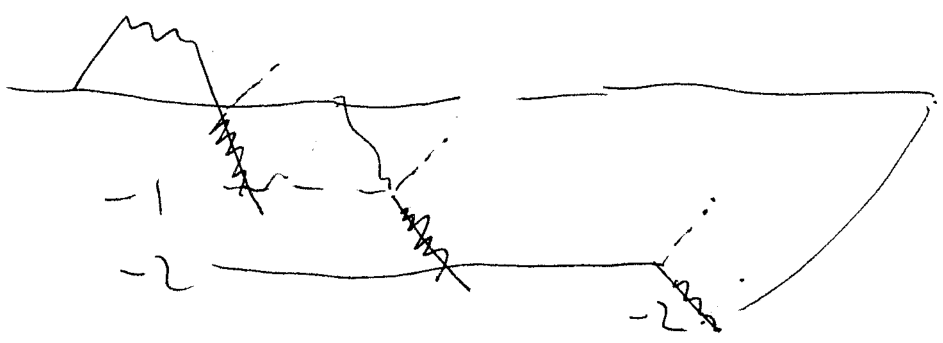


$n = 2k$ $\binom{n}{\lfloor n/2 \rfloor}$





↖ clearly $\binom{n}{\lfloor n/2 \rfloor}$ # objects like this is



symmetry:
$a_i = a_{N-i}$

unimodal: $a_0 \le a_1 \le \cdots \le a_{\lfloor N/2 \rfloor} \geq \cdots \geq a_N$

10/19/06

At most 6 problems. Few facts can use for PSET 3.

Can use following facts: $s_{\lambda/\mu} = \sum_{T\ \mathrm{SSYT}(\lambda/\mu)} x^{\mathrm{weight}(T)} \in \Lambda$ ring of symmetric fncts

Involution $\omega : \Lambda \to \Lambda$ homomorphism, $\omega : s_{\lambda/\mu} \longmapsto s_{(\lambda/\mu)'}$

Jacobi-Trudi identity $\lambda = (\lambda_1, \dots, \lambda_n)$, $s_\lambda = \det(h_{\lambda_i - i + j})_{i,j=1}^{n}$

$h_0 = 1$, $h_k = 0$ $k < 0$, $h_k = \sum_{i_1 \le i_2 \le \cdots \le i_k} x_{i_1} \cdots x_{i_k}$ complete homogeneous symm fnct

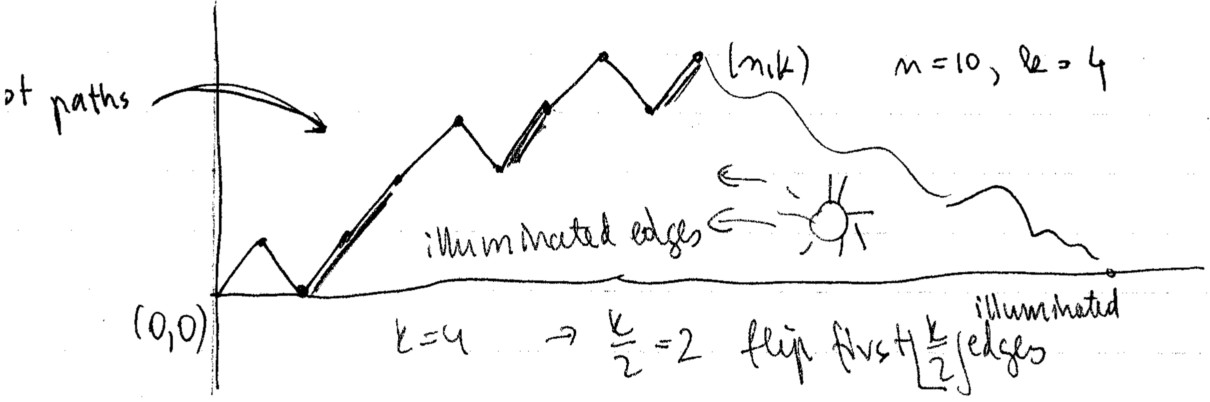
$$s_{\lambda/\mu} = \det(h_{\lambda_i - i - \mu_j + j})_{i,j=1}^{n}$$

$$s_{311} = \begin{vmatrix} h_3 & h_4 & h_5 \\ h_0 = 1 & h_1 & h_2 \\ 0 & h_0 = 1 & h_1 \end{vmatrix}$$

Classical defn of s_λ: $s_\lambda(x_1, \dots, x_n) = \dfrac{\det(x_j^{\lambda_i + n - i})_{i,j=1}^{n}}{\prod_{i<j} (x_i - x_j)}$

$$s_{311}(x_1, x_2, x_3) = \frac{\begin{vmatrix} x^5 & x_2^5 & x_3^5 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \end{vmatrix}}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}$$

PSET 2, solutions. ① a) $\overbrace{\begin{array}{|ccc|cccc|}\hline 1&3&4&5&7&8&10\\ \hline 2&6&9\\ \cline{1-3}\end{array}}^{n}$ $\underbrace{\hphantom{5\ 7\ 8\ 10}}_{k}$ ↗ ↘ $\sum_{\lambda=(\lambda_1,\lambda_2)} (f^\lambda)^2$



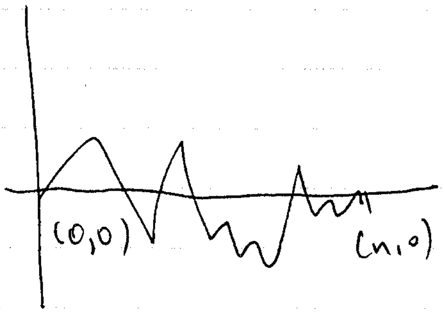
$k=4 \rightarrow \frac{k}{2}=2$ flip first $\lfloor\frac{k}{2}\rfloor$ illuminated edges

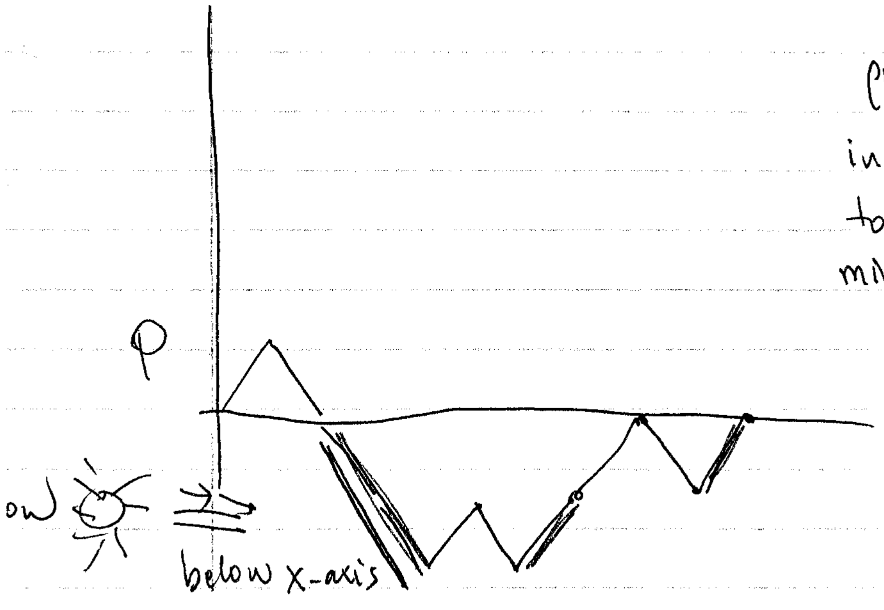
pairs → Catalan #

b) $\sum_{\lambda=(\lambda_1,\lambda_2)} f^\lambda = \binom{n}{\lfloor\frac{n}{2}\rfloor} \longrightarrow$ # paths from (0,0) → (n,0)

n even ↗

n odd → (n,1)

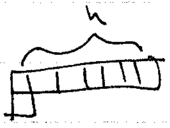




flip over.

Claim. Ballot paths from (0,0) to (n,k) are in bijection w/ all lattice paths from (0,0) to (n,0) (or (n,1) if n odd) s.t. minimum of P = $-\lfloor\frac{k}{2}\rfloor$.

② (n-1)-dim'l repn is $\simeq V_{(n,1)}$



$u_i = (1 0 .. 0 . -1 0 ... 0)$ $\{u_i\}$ are basis of

↑ith

$\mathcal{B} = \{\sum_{i=2}^{k} u_i\}_{k=2}^{n}$

Lecho

$v_1 = (1\ -1\ 0\ ..\ )$

$v_2 = (1\ 1\ -2\ 0 .\ )$

$v_3 = (1\ 1\ 1\ -3\ 0 ..\ )$

$v_4 = (1\ 1\ 1\ 1\ -4\ 0 ..)$

X_4 actym v_3

$$\begin{array}{lcccc} (14) & -3 & 1 & 1 & 1\\ (24) & 1 & -3 & 1 & 1\\ (34) & 1 & 1 & -3 & 1\\ \hline & -1 & -1 & -1 & 3\end{array}$$

de Bruin

$$\begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & \\ x_1 & 0 & 0 & 0 & 0 & 0 & \\ x_2 & -1 & 1 & 1 & 1 & 1 & 1 \\ x_3 & 1 & -1 & 2 & 2 & 2 & 2 \\ x_4 & 2 & 2 & -1 & 3 & 3 & 3 \\ & 3 & 3 & 3 & & & \end{array}$$

$I \subseteq [n-1]$, $S(I) = \{ j \mid |\{j, j+1\} \cap I| = 1 \}$

$S(\{1, 3, 4, 6\}) = \{1, 2, 4, 5, 6\}$

$\beta(I)$ = # perm w/ descent set I

<u>PS3</u>: ⑦ if $S(I) \supseteq S(J)$ then $\beta(I) \geq \beta(J)$

⑧ $(x_1 + x_2 + \ )^n = \sum\limits_{\kappa - n\text{-ribbon}} s_\kappa$

⑨

⑩ $K_{\lambda\mu} \leq K_{\lambda\nu}$, when $\mu \geq \nu$ in dominance order.

<u>Symmetric functions</u>

<u>10/24/06</u>

$\Lambda = \varprojlim \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ (in the category of graded rings)

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ partition $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$

$m_\lambda = \sum\limits_{\substack{i_1 \ldots i_k \\ \text{distinct}}} x_{i_1}^{\lambda_1} \ldots x_{i_k}^{\lambda_k}$ monomial symm fncts

$m_2 = x_1^2 + x_2^2 + \ldots$

$m_{11} = \sum\limits_{i<j} x_i x_j$

$m_{21} = \sum\limits_{i<j} (x_i^2 x_j + x_i x_j^2)$

$e_k = m_{1^k}$, $h_k = \sum_{|\lambda| \vdash k} m_\lambda$, $p_k = m_{(k)} = x_1^k + x_2^k + \ldots$

$\uparrow$ elementary $\uparrow$ complete homog. $\uparrow$ power symm fnct

$e_\lambda = e_{\lambda_1} \ldots e_{\lambda_k}$ $h_\lambda = h_{\lambda_1} \ldots h_{\lambda_k}$ $p_\lambda = p_{\lambda_1} \ldots p_{\lambda_k}$

Fundamental Theorem of Symm Fncts (FTSF) $\Lambda = \mathbb{Z}[\underset{\uparrow}{e_1}, \underset{\nearrow}{e_2}, e_3, \ldots]$

alg. indep.

Similarly, $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$

$$\Lambda \neq \mathbb{Z}[p_1, p_2, \ldots]$$

$$\underbrace{\Lambda \otimes \mathbb{Q}}_{\text{symm fnct w/ rat. coeff}} = \mathbb{Q}[p_1, p_2, \ldots]$$

<u>Theorem.</u> Λ has $\mathbb{Z}$-basis (as a linear space) $\{m_\lambda\}$, $\{e_\lambda\}$, $\{h_\lambda\}$

and $\mathbb{Q}$-basis $\{p_\lambda\}$

<u>Claim.</u> dim of Λ^k (k-th graded component of Λ) $= p(k) = \#$ partitions of k

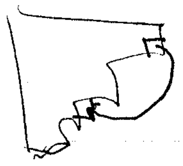
<u>Dominance order on partitions</u>

$\lambda, \mu \quad : \quad |\lambda| = |\mu|$

$\lambda \geq \mu \iff \lambda_1 \geq \mu_1, \quad \lambda_1 + \lambda_2 \geq \mu_1 + \mu_2$

<u>Claim</u> dominance order is generated by following relations:

$$(\lambda_1, \lambda_2, \ldots \lambda_n) > (\lambda_1 \ldots \lambda_i - 1, \ldots \lambda_j + 1 \ldots) \qquad i < j$$



get something less in dominance order.

Corollary. $\lambda \geq \mu \iff \lambda' \leq \mu'$

Lemma. $e_{\lambda'} = m_\lambda + \sum_{\mu < \lambda} a_{\lambda\mu} m_\mu \quad , \quad a_{\lambda\mu} \in \mathbb{Z}.$

Dominance order is compatible with lexicographic order.

Proof. $$e_{3,2} = \left(\sum_{i<j<k} x_i x_j x_k\right)\left(\sum_{l<m} x_l x_m\right) = (x_1x_2x_3)(x_1x_2) + \cdots$$

$$= m_{(3,2)'} + \sum_{\underset{\uparrow}{\mu < (3,2)'}} a_{\lambda\mu} m_\mu$$

in lexicographic

but can assume this is dominance order

Lemma. (1) $$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = \begin{cases} 0 & n > 0 \\ 1 & n = 0 \end{cases}$$

(2) $$n h_n = \sum_{r=1}^{n} p_r h_{n-r}$$

(2)' $$n e_n = \sum_{r=1}^{n} (-1)^r p_r e_{n-r}$$

Proof. $$E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t)$$

$$H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} \frac{1}{(1 - x_i t)}$$

$$E(t) H(-t) = 1 \iff (1)$$

$$P(t) = \sum_{r \geq 1} p_r t^{r-1} = \sum_{i, r \geq 1} x_i^r t^{r-1} = \sum_{i \geq 1} \frac{x_i}{1 - x_i t} = \frac{d}{dt}\left(\sum_{i \geq 1} \log\left(\frac{1}{1 - x_i t}\right)\right)$$

$$= \frac{d}{dt} \log \prod_{i \geq 1} \frac{1}{1 - x_i t} = \frac{d}{dt}\left(\log H(t)\right) = \frac{H'(t)}{H(t)} \Rightarrow H'(t) = P(t) H(t) \overset{(2)}{\Leftarrow}.$$

Proof of Theorem. $\{m_\lambda\}$ is a $\mathbb{Z}$-basis of Λ by definition

$\{e_{\lambda'}\}$ related to $\{m_\lambda\}$ by a triangular transformation

$\{e_{\lambda'}\}$ is expressed thru basis m_λ by triangular matrix w/ 1's on diag s.t.

$$e_{\lambda'} = m_\lambda + \sum_{\mu < \lambda} a_{\lambda\mu} m_\mu$$

$\Rightarrow$ $\{e_\lambda\}$ is a $\mathbb{Z}$-basis.

$$(1): \quad h_n - \underbrace{h_{n-1} e_1 + h_{n-2} e_2 - \cdots}_{\text{expressed} \to \text{ express } h_n} = 0$$

so h_n can be expressed as a polynomial in $e_1, e_2, \ldots, e_n$ w/ integer coeffs. and vice versa.

$\Rightarrow$ $\{h_\lambda\}$ is a $\mathbb{Z}$-basis.

h_n can be expressed in terms of $p_1, \ldots, p_n$ with coeffs in $\mathbb{Q}$ and vice versa (by (2))

$\Rightarrow$ $\{p_\lambda\}$ is a $\mathbb{Q}$- basis of Λ.

Schur functions

$$s_\lambda = \sum_{\beta\text{-compositions}} K_{\lambda\beta} x^\beta \quad , \quad K_{\lambda\beta} = \#SSYT(\lambda) \text{ weight } \beta$$

$$= \sum_{\mu \text{ partitions}} K_{\lambda\mu} m_\mu$$

Lemma. For two partitions λ, μ , $|\lambda| = |\mu|$ (1) $K_{\lambda\lambda} = 1$ (2) if $K_{\lambda\mu} \neq 0$ $\Rightarrow \mu \leq \lambda$ in dominance order

Proof.

$$T_0 = \begin{array}{|cccccccc}1&1&1&1&1&1&1&1\\2&2&2&2&2\\3&3&3\\4&4\end{array}$$

$K_{\lambda\lambda} = 1$ trivially.

any other SSYT of shape λ is obtained from T_0 by swaps of some i's w/ j's, $i<j$.

just chage. not bother any other enty

Every time substract 1 from i-th component of weight & add 1 to j-th $\to$

$$\Rightarrow \mu \le \lambda.$$

s_λ is also a $\mathbb{Z}$-basis, since matrix between s_λ, m_λ is upper triangular.

Theorem. $\{s_\lambda\}$ is a $\mathbb{Z}$-basis of Λ.

Proof. The Kostka matrix $(K_{\lambda\mu})$ is upper triangular, with 1's on the diagonal.

In particular it is invertible. etc.

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$$

heorem.

$$h_\lambda = \sum_\mu K_{\mu\lambda} s_\mu$$. To see this we introduce inner product.

Scalar product on Λ.

Claim. $$\prod_{i,j\ge 1} \frac{1}{1-x_iy_j} \overset{\text{Cauchy}}{=} \sum_\lambda s_\lambda(x)\cdot s_\lambda(y) = \sum_\lambda h_\lambda(x)\, m_\lambda(y)$$

Proof. (1) Cauchy

(2) $$\prod \frac{1}{1-x_iy_j} = \prod_j H(y_j) = \prod_{j\ge 1}\Big(\sum_{r\ge 0} h_r(x)\, y_j^r\Big) = \sum_{\alpha\text{-composition}} \overset{h_{\alpha_1}h_{\alpha_2}\cdots}{\overset{\parallel}{h_\alpha(x)}}\ \overset{y_1^{\alpha_1}y_2^{\alpha_2}\cdots}{\overset{\parallel}{y^\alpha}} =$$

$$= \sum_{\lambda \text{ partition}} h_\lambda(x)\, m_\lambda(y) \quad ▨$$

Define scalar product on Λ by $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$, for $\{h_\lambda\}, \{m_\mu\}$ are dual bases.

Lemma Two bases of Λ $\{u_\lambda\}$, $\{v_\mu\}$
TFAE:
(1) $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$
(2) $\sum u_\lambda(x)\, v_\lambda(y) = \prod \frac{1}{1-x_i y_j}$

PROOF: $u_\lambda = \sum_\rho a_{\lambda\rho} h_\rho$, $v_\mu = \sum_\sigma b_{\mu\sigma} m_\sigma$

$(1) \Leftrightarrow \sum_\rho a_{\lambda\rho} \cdot b_{\mu\rho} = \delta_{\lambda\mu} \Leftrightarrow$
$A \cdot B^T = I$

$$(2) \Leftrightarrow \sum_\lambda u_\lambda(x)\, v_\lambda(y) = \sum_\rho h_\rho(x)\, m_\rho(y) \Leftrightarrow \sum_\lambda a_{\lambda\rho} b_{\lambda\sigma} = \delta_{\rho\sigma} \Leftrightarrow$$

$$A^T B = I . \Leftrightarrow (1) \quad . ▨$$

Corollary: $\{s_\lambda\}$ is an orthogonal basis of Λ.

Dualize $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$ you get $h_\lambda = \sum_\mu K_{\mu\lambda} s_\mu$.

Involution ω $\omega : \Lambda \to \Lambda$ homomorphism given by $\omega(h_k) = e_k$.

Theorem. (1) $\omega(h_\lambda) = e_\lambda$
(2) $\omega(e_\lambda) = h_\lambda$
(3) $\omega(s_\lambda) = s_{\lambda'}$

Proof. (1) by defn. (3) $\sum_\lambda s_\lambda(x)\, s_\lambda(y) = \prod \frac{1}{1-x_i y_j} = \prod H(y_j) = \sum_\lambda h_\lambda(x)\, m_\lambda(y)$

Apply $\underline{\omega}_{\textcircled{x}}$ (that depend on x)

$$\sum_\lambda \omega(s_\lambda(x))\, s_\lambda(y) = \sum_\lambda \overbrace{\omega(h_\lambda(x))}^{e_\lambda(x)}\, m_\lambda(y) = \prod E(y_j)$$

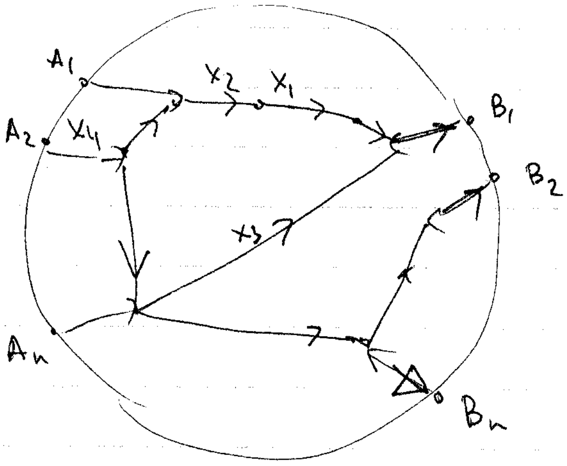
$$= \prod_{i,j \geq 1} (1 + x_i y_j) = \underbrace{\sum_\lambda s_{\lambda'}(x)\, \underbrace{s_\lambda(y)}_{\text{form basis}}}_{\text{dual Cauchy}} \Rightarrow \boxed{s_{\lambda'}(x) = \omega(s_\lambda(x))}$$

From (3) $\omega(s_\lambda) = s_{\lambda'} \Rightarrow \omega^2(s_\lambda) = s_\lambda \Rightarrow \omega$ involution $\Rightarrow$ (1) $\Rightarrow \omega(e_\lambda) = h_\lambda$.

also e_k - special case of Schur funct, and h_k - special case of Schur funct.

Lindström Lemma (aka Gessel-Viennot method)

G is a planar acyclic digraph with sources $A_1, .., A_n$ (on the left) and sinks $B_1, ..., B_n$ (on the right) with weights assigned to edges.



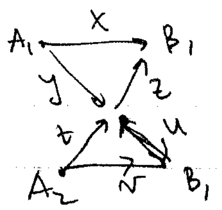
Let $M_{ij} = \sum_{p: A_i \to B_j} \prod_{e \in p} \text{weight}(e)$

$M = (M_{ij})$ $n \times n$ matrix

<u>indström lemma.</u> $\det(M) = \sum_{\substack{p_1: A_1 \to B_1 \\ p_2: A_2 \to B_2 \\ \dots \\ p_n: A_n \to B_n}} \prod_{i=1}^{n} \prod_{e \in p_i} \text{weight}(e)$

$p_1, .., p_n$ are noncrossing paths

Ex.

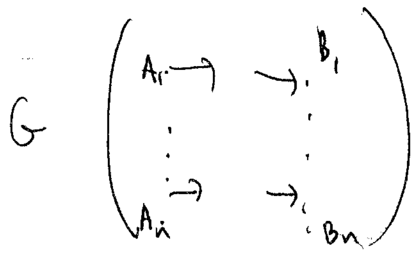


$$\begin{vmatrix} x + yz & yu \\ tz & tu + v \end{vmatrix} = x \cdot v + yz \cdot v + x \cdot tu$$

Lindström's Lemma.

10/26/06

G planar digraph



$$M_{ij} = \sum_{P: A_i \to B_j} \text{weight}(P)$$

$$\det(M_{ij}) = \sum_{\substack{P_1: A_1 \to B_1 \\ \vdots \\ P_n: A_n \to B_n \\ \text{noncrossing}}} \prod_i \text{weight}(P_i)$$

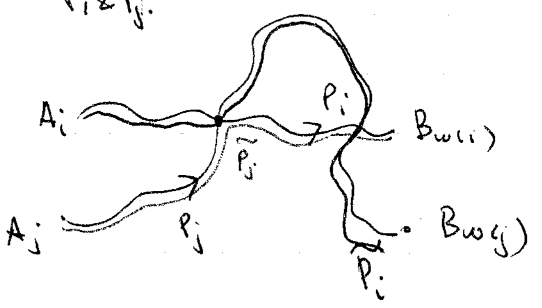
(no common vertices

Proof.

$$\det M = \sum_{w \in S_n} (-1)^w \sum_{\substack{P_1: A_1 \to B_{w(1)} \\ P_2: A_2 \to B_{w(2)} \\ \vdots \\ P_n: A_n \to B_{w(n)}}} \prod \text{weight}(P_i)$$

Involution principle $P = (P_1, .., P_n)$ collection of paths with an intersection.

Find lexicographically minimal pair (i,j) s.t. $P_i \cap P_j \neq \emptyset$. Let x = first common pt of P_i & P_j.



Let $\tilde{P}_i$, $\tilde{P}_j$ be two paths obtained from P_i & P_j by swapping their tales at x.

Define map $\iota: P \to (P_1, .., \tilde{P}_i, .. \tilde{P}_j, .. P_n) =: \tilde{P}$

1) ι is an involution

2) ι changes the sign of w, so contribution of path P cancels contribution of $\tilde{P}$ →

we can disregard crossing paths in formula for determinant

If all weights in our graph G are nonnegative ≥ 0, then M is a totally nonnegative matrix (all minors are nonnegative)

$$\begin{pmatrix} a & b & c & .. \\ d & e & f & .. \\ - & - & - & \end{pmatrix} \quad a, b, c, d, e, f, .. \geq 0 \quad , \quad \begin{vmatrix} a & b \\ d & e \end{vmatrix}, \begin{vmatrix} a & c \\ d & f \end{vmatrix}, .. \geq 0$$

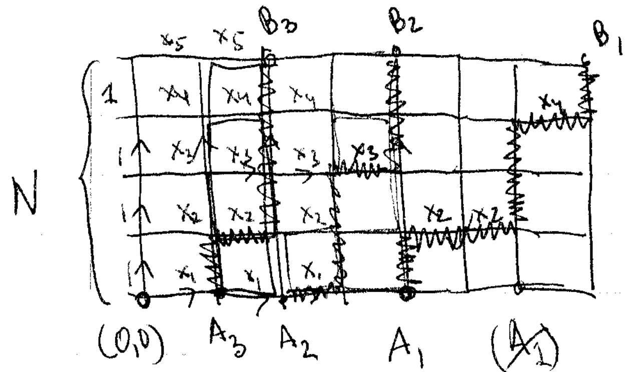
etc. all minors ≥ 0 . . .

More subtle result: Inverse claim: Each TNM (totally nonnegative matrix) comes from some planar graph with nonnegative weights.

Jacobi-Trudi identity

$\lambda=(\lambda_1,\dots,\lambda_n)$, $\mu=(\mu_1,\dots,\mu_n)$ $\lambda \supseteq \mu$

$$s_{\lambda/\mu} = \det\left(h_{\lambda_i-i-\mu_j+j}\right)_{i,j=1}^{n}$$



if $\mu=(4,1,1)$

all vertical edges weight 1

$$N-1$$
$$\downarrow$$

$A_i=(\mu_i+n-i,0)$ $B_j=(\lambda_j+n-j, N-1)$

$\lambda=(5,3,2)$ $N=5$

$\mu=(2,1,1)$ $n=3$

N will be # variables. Will let it go to ∞

$M_{33}=x_1+x_2+x_3+x_4+x_5 = h_1(x_1,\dots,x_N)$

$M_{22}=x_1x_1+x_1x_2+x_2x_2+\cdots = h_2(x_1,\dots,x_N)$

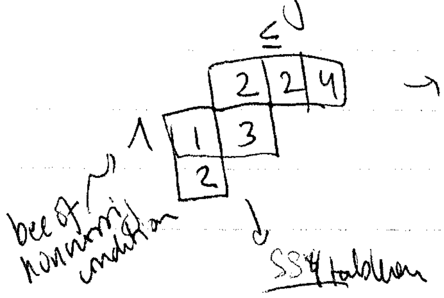
$M_{ij}=h_{\lambda_j-j-\mu_i+i}(x_1,\dots,x_N)$

Lindström lemma ⇒ JT-det = $\det(M_{ij}) = \sum\limits_{\substack{P \text{ noncrossing}\\ \text{family of paths}}} \text{weight}(P)$

Claim. {Families of Noncrossing paths} ⟷ SSYT's of shape λ/μ

and weight of tableau ⟷ weight of path.

$\lambda/\mu=(5,3,2)/(2,1,1)$



$\to A_1 \to B_1$ $x_2x_2x_4$

$A_2 \to B_1$ x_1x_3

$A_3 \to B_3$ x_2

P noncrossing $\Leftrightarrow$ tableau's columns are strictly increasing

So >T det $= \det(M_{ij}) = \sum_{\substack{P \text{ noncrossing} \\ \text{family of paths}}} \text{weight}(P) = s_{\lambda/\mu}(x_1, .., x_N)$.

Take limit as $N \to \infty$ → get Jacobi-Trudi identity.

Dual Jacobi-Trudi

$$s_{(\lambda/\mu)'} = \det\left(e_{\lambda_i - i - \mu_j + j}\right)_{i,j=1}^{n}$$ by application of ω involution

<u>Classical defn of Schur functions</u>: $\alpha = \alpha_1 > \alpha_2 > .. > \alpha_n$

$$a_\alpha = \det(x_i^{\alpha_j})_{i,j=1}^{n} = \begin{vmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} & \dots & x_n^{\alpha_1} \\ x_1^{\alpha_2} & x_2^{\alpha_2} & & \\ & & & \\ x_1^{\alpha_n} & x_2^{\alpha_n} & & x_n^{\alpha_n} \end{vmatrix}$$ generalized van der Monde determinant

a_α is divisible by $x_i - x_j$ $\Rightarrow$ a_α is divisible by $\prod_{i<j}(x_i - x_j)$

<u>Ex.</u> $\rho = (n-1, n-2, \dots, 0)$ also denoted by δ

$a_\rho = \prod_{i<j}(x_i - x_j)$

divisible & has same degree.

weight, ½·sum of weyl roots

<u>Theorem</u>. $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$ partition

$$\left(S_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}\right) \qquad S_\lambda(x_1, .., x_n) = \frac{a_{\lambda+\rho}}{a_\rho}$$ ⟵ formally equiv. to weyl char formula in type A

<u>a</u>. Let $\beta = (\beta_1, .., \beta_n)$ composition. $A_\beta = (x_j^{\beta_i})$, $H_\beta = (h_{\beta_i - n + j})$, ← in first n variables $x_1 .. x_n$.

$E = ((-1)^{n-i} e_{n-i}^{(j)})$ where $e_k^{(j)} = e_k(x_1, .., x_{j-1}, x_{j+1}, \dots x_n)$ skip jth variable.

Then $A_\beta = H_\beta E$

<u>Proof.</u> $E^{(j)}(t) = \sum_{k\geq 0} e_k^{(j)} t^k = \prod_{\substack{i\neq j\\ i=1..n}} (1+x_i t)$

$$H(t) = \sum_{k\geq 0} h_k t^k = \prod_i \frac{1}{1-x_i t}$$

$$H(t)\,E^{(j)}(-t) = \frac{1}{1-x_j t}$$

extract coeff of t^{β_i}

$$\sum h_{\beta_i-\ell} (-1)^\ell e_\ell^{(j)} = x_j^{\beta_i}$$

$$\sum h_{\beta_i-n+r} (-1)^{n-r} e_{n-r}^{(j)} = x_j^{\beta_i}$$

$$\Downarrow$$

$$H_\beta \cdot E = A_\beta.$$

<u>Proof of Theorem.</u> $\underset{\overset{\parallel}{a_\beta}}{|A_\beta|} = |H_\beta||E|$

Take $\beta = \rho$ - Then $a_\rho = \overset{\det = 1}{\begin{vmatrix} 1 & h_1 & h_2 & & \\ & 1 & h_1 & h_2 & \\ & & 1 & h_1 & h_2 \\ & 0 & & 1 & \\ & & & & 1 \end{vmatrix}} |E| \Rightarrow |E| = a_\rho$!

Take $\beta = \lambda + \rho$

$$a_{\lambda+\rho} = \underset{\substack{\uparrow \\ \text{J-T-det for } s_\lambda \\ = s_\lambda}}{|H_{\lambda+\rho}|} - \underset{\substack{\parallel \\ |E|.}}{a_\rho}. \quad \boxtimes$$

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}. \quad \boxtimes$$

Determinantal formula for $f^{\lambda/\mu}$ = #SYT of shape λ/μ.

__Theorem.__ Pick $\lambda=(\lambda_1,\ldots,\lambda_n)$, $\mu=(\mu_1\ldots\mu_n)$, $\lambda\geq\mu$, $N=|\lambda/\mu|$.

$$f^{\lambda/\mu} = N!\,\det\left(\frac{1}{(\lambda_i-i-\mu_j+j)!}\right)_{i,j=1}^{n}, \quad 0!=1, \quad \boxed{\text{so } \frac{1}{k!}=0} \quad k<0.$$

__Example.__ 

$\lambda=(3,2)$, $\mu=(1,0)$

$$f^{\lambda/\mu}=4!\begin{vmatrix}\frac{1}{2!} & \frac{1}{4!}\\ \frac{1}{0!} & \frac{1}{2!}\end{vmatrix} = 4!\left(\frac{1}{2}\cdot\frac{1}{2}-\frac{1}{2\cdot3\cdot4}\right)=5$$

$\#\{w_1<w_2>w_3<w_4\}=5$
$\{w_1,w_2,w_3,w_4\}$

Instead of $\underset{\substack{\uparrow\\ \text{semistand}}}{s_{\lambda/\mu}} \rightsquigarrow \underset{\substack{\uparrow\\ \text{standard.}}}{f^{\lambda/\mu}}$

$f^{\lambda/\mu}=[x_1x_2\cdots x_n]\,s_{\lambda/\mu}$ ← coeff of $x_1,\ldots,x_N$

Exponential specialization $\Lambda_{\mathbb{Q}}=\Lambda\otimes\mathbb{Q}$ symm fnct w/ rational coeffs.

$ex:\Lambda_{\mathbb{Q}}\to\mathbb{Q}[t]$

$\Lambda_{\mathbb{Q}}=\mathbb{Q}[p_1,p_2,\ldots]$

$g(p_1,p_2,\ldots)\mapsto g(t,0,0,\ldots)$

__Lemma:__ $f(x_1,x_2,\ldots)\in\Lambda_{\mathbb{Q}}$, then $ex(f)=\sum_{n\geq0}[x_1\cdots x_{\text{[illegible]}}]f\cdot\frac{t^n}{n!}$.

__Proof.__ Enough to prove for $p_\lambda=p_{\lambda_1}p_{\lambda_2}\cdots$ $\quad ex(p_\lambda)=\begin{cases}t^n & \text{if } \lambda=1^n\\ 0 & \text{o.w.}\end{cases}$

$$[x_1 \cdots x_n]\ (x_1^{\lambda_1} + x_2^{\lambda_1} + \cdots)(x_1^{\lambda_2} + x_2^{\lambda_2} + \cdots) = \begin{cases} n! & \text{if } \lambda = 1^n \\ 0 & \text{o.w.} \end{cases}$$

So lemma true.

$$ex(s_{\lambda/\mu}) = [x_1 \cdots x_N]\, s_{\lambda/\mu} \frac{t^N}{N!} = f^{\lambda/\mu} \frac{t^N}{N!}$$

$$ex(h_k) = \frac{t^k}{k!}$$

$$s_{\lambda/\mu} = \det(h_{\lambda_i - i - \mu_j + j})$$

ex $\downarrow$

$$\frac{f^{\lambda/\mu}}{N!} t^N = \det\left(\frac{1}{(\lambda_i - i - \mu_j + j)!}\right) t^N \quad \boxtimes$$

Corollary

$f^\lambda = N! \det\left(\frac{1}{(\lambda_i - i + j)!}\right)_{i,j=1}^N$ In this case $\det(\ ) = \dfrac{1}{\prod_{x \in \lambda} h(x)}$.

{Symmetric functs} $\leftrightarrow$ {Rep of S_n} 10/31/06

R^n = space of class functs on S_n $(f(w) = f(uwu^{-1})\ \forall u,w \in S_n)$

Bases $\{\chi_\lambda\}$ char of reps

$\{c_\mu\}$ $c_\mu(w) = \begin{cases} 1 & \text{if } \rho(w) = \mu \\ 0 & \text{o.w.} \end{cases}$ cycle type

Λ^n - homogeneous symm functs of degree n / $\mathbb{C}$

Frobenius characteristic map $ch: R^n \to \Lambda^n$

$$ch: f \mapsto \frac{1}{n!} \sum_{w \in S_n} f(w)\, p_{\rho(w)}$$

$$= \sum_{\mu \vdash n} z_\mu^{-1} f(\mu)\, p_\mu$$

$$\frac{n!}{z_\mu} = \# \{ w \in S_n \mid \rho(w) = \mu \}$$

Lemma. G finite group, $g \in G$, C - conjugacy class of g

$H = \{ h \in G \mid hgh^{-1} = g \}$

Then $|G| = \frac{|G|}{|H|}$.

Proof. G acts on itself by conjugations. $h : g \mapsto hgh^{-1}$

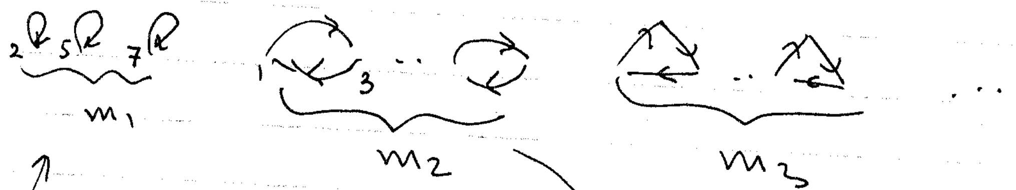
C - orbit of g

H is the stabilizer subgroup of g. $\Rightarrow$ $|G| = |G|/|H|$.

$$\frac{n!}{z_\mu} = \frac{n!}{\# \{ u \in S_n \mid u w_\mu u^{-1} = w_\mu \}}$$ (Pick an elt w_μ, $\rho(w_\mu) = \mu$.

Let $\mu = (1^{m_1} 2^{m_2} 3^{m_3} \ldots)$

w_μ -



u has to send any fixed point to fixed..

preserve cycles & cyclicly shift cycles.

$$z_\mu = 1^{m_1} m_1! \, 2^{m_2} m_2! \, 3^{m_3} m_3! \ldots$$

cyclic shift

perm of parts of equal parts

__Frobenius Theorem.__ ch is an isomorphism of linear spaces.

(1) $\mathrm{ch}(C_\lambda) = z_\lambda^{-1} p_\lambda$ (by the definition)

(2) $\mathrm{ch}(\chi_\lambda) = s_\lambda$

(3) ch preserves the inner product.

__q-analog__ (of determinantal formula?)

Specialization $\Lambda \to \mathbb{Z}[[q]]$ ← ring of formal power series

$x_i \mapsto q^{i-1}$

$$s_{\lambda/\mu} \longmapsto s_{\lambda/\mu}(1, q, q^2, \dots) = \sum_{T-\mathrm{SSYT}(\lambda/\mu)} q^{\sum_{ij}(T_{ij}-1)}$$

T_{ij} – (i,j) entry of

$$h_k(1, q, q^2, \dots) = \sum_{1 \le i_1 \le i_2 \le \dots \le i_k} q^{(i_1-1)+(i_2-1)+\dots+(i_k-1)}$$

$$= \sum_{j_1, \dots, j_k \geq 0} q^{j_1 + (j_1+j_2) + (j_1+j_2+j_3) + \dots} = \sum q^{kj_1 + (k-1)j_2 + \dots + j_k} =$$

$$= \prod_{l=1}^{k} \left(\sum_{j \geq 0} q^{lj}\right) = \frac{1}{1-q}\,\frac{1}{1-q^2} \cdots \frac{1}{1-q^k} = \frac{1}{(1-q)^k}\,\frac{1}{[k]_q!}$$

J-T ⇒ __Corollary.__ $s_{\lambda/\mu}(1, q, q^2, \dots) = \cancel{\det\left(\frac{1}{(1-q)^{\lambda_i-i-\mu_j+j}\,[\lambda_i-i-\mu_j+j]_q!}\right)}$

$$= \det\left(\dots = \frac{1}{(1-q)^{|\lambda/\mu|}} \det\left(\frac{1}{[\lambda_i - i - \mu_j + j]_q!}\right)_{i,j=1}^{\#\text{rows in } \lambda}\right.$$

Last time we had $f^{\lambda/\mu} = \det\left(\frac{1}{(\lambda_i - i - \mu_j + j)!}\right)$

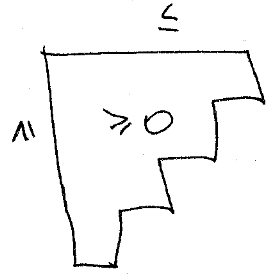
Theorem (Stanley) $s_\lambda(1, q, q^2, \ldots) = q^{n(\lambda)} \prod_{x \in \lambda} \frac{1}{1 - q^{h(x)}}$

$n(\lambda) = \sum_i (i-1)\lambda_i$

$$\begin{array}{|cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 \\
3 & 3
\end{array}$$

Def. Reverse plane partitions (RPP) of shape λ

T =



all rows & columns weakly increasing

$$\begin{array}{|cccccc}
0 & 0 & 1 & 3 & 5 & 5 \\
0 & 1 & 2 & 3 & 5 \\
1 & 1 & 4 \\
3 & 4
\end{array}$$

{RPP}, in bijection w/ SSYT

SSYT $\rightarrow$ RPP just substract i from elts of the ith row.

Theorem. $\sum_{T \in RPP(\lambda)} q^{\sum T_{ij}} = \prod_{x \in \lambda} \frac{1}{1 - q^{h(x)}}$ (since bijection above just killed $q^{n(\lambda)}$)

RHS $= \sum_{F} q^{\sum F_{ij} h(i,j)}$

$F \leftarrow$ any function from box of λ to $\mathbb{Z}_{\geq 0}$

Hillman-Grassl correspondence. HG: {RPP's of shape λ} $\longleftrightarrow$ {functions on boxes of λ}

bijection

$$T = \begin{array}{|cccc}
1 & 2 & 2 & 2 \\
3 & 3 & 3 \\
3
\end{array} \longleftrightarrow \begin{array}{|cccc}
2 & 0 & 0 & 0 \\
1 & 1 & 1 \\
0
\end{array} = F$$

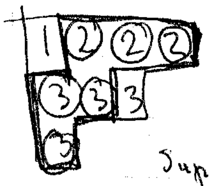
sum of entries of T = $2 \cdot 6 + 1 \cdot 4 + 1 \cdot 2 = \sum$ entry × hook length

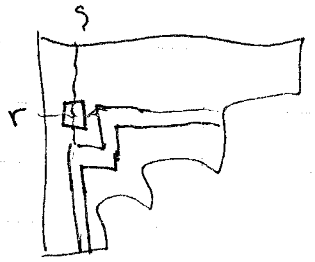
T- RPP $\rightsquigarrow$ F

Construct the ribbon path p in T s.t.

(1) p starts in the most north east box (a,b) s.t. $T_{ab} \neq 0$

(2) $(i,j) \in p \Rightarrow \begin{cases} (i, j-1) \in P \text{ if } T_{i,j-1} = T_{i,j} \\ (i+1, j) \in P \text{ if } T_{i,j-1} < T_{i,j} \\ \qquad\qquad (i+1, j) \in \lambda \\ \text{Stop} \quad \text{o.w.} \end{cases}$



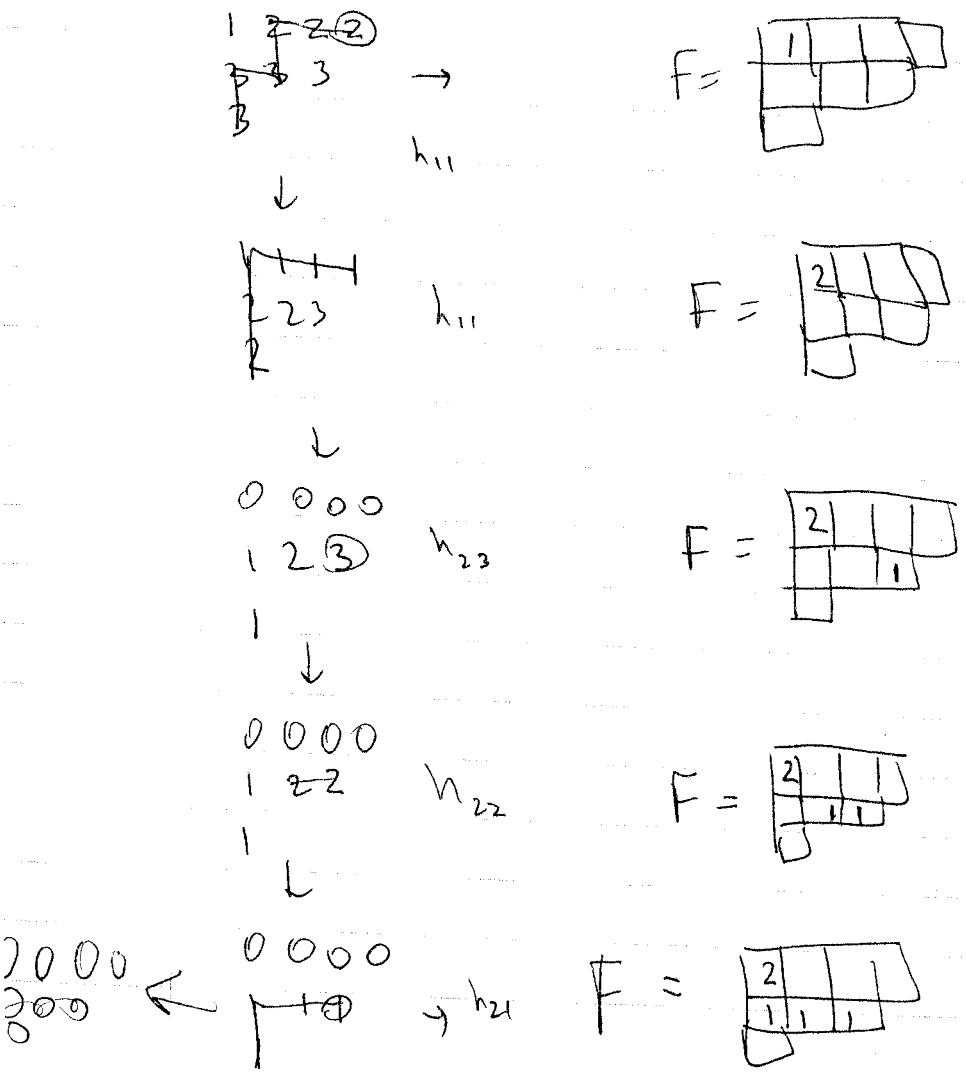


Suppose that p starts at the r-th row & ends at the s-th column.

Then (1) add 1 to F(r,s)

(2) substract 1's from all elts of P.

Repeat this procedure until we get ~~tableau~~ RPP w/ all 0's.



Need to prove that this is a bijection. So need to show inverse.

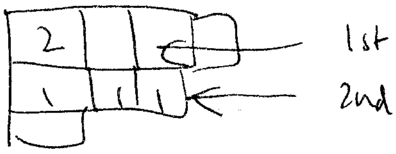
So need to figure out how to go from function on boxes F to tableau T.

We "decompose" T into hook-length $(h_{11}, h_{11}, h_{23}, h_{22}, h_{21})$

Lemma. In this decomposition h_{ij} occurs before $h_{i'j'}$ iff

$i<i'$ or $i=i, j \geq j'$

F =



$h_{11}, h_{11}, h_{23}, h_{22}, h_{21}$

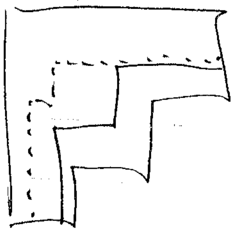
Proof of lemma. $T \xrightarrow[p]{} T' \xrightarrow[p']{} T''$

p corresponds to h_{ij}
p' — " — $h_{i'j'}$

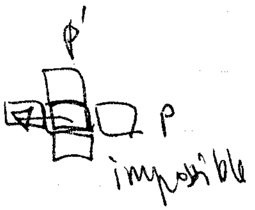
Need $i<i'$ or $i=i'$ & $j \geq j'$.

It is clear that $i \leq i'$. If $i<i'$ then we are done. So assume $i=i'$.

Claim. If $i=i'$ then p' is weakly above p.



Suppose not.



impossible

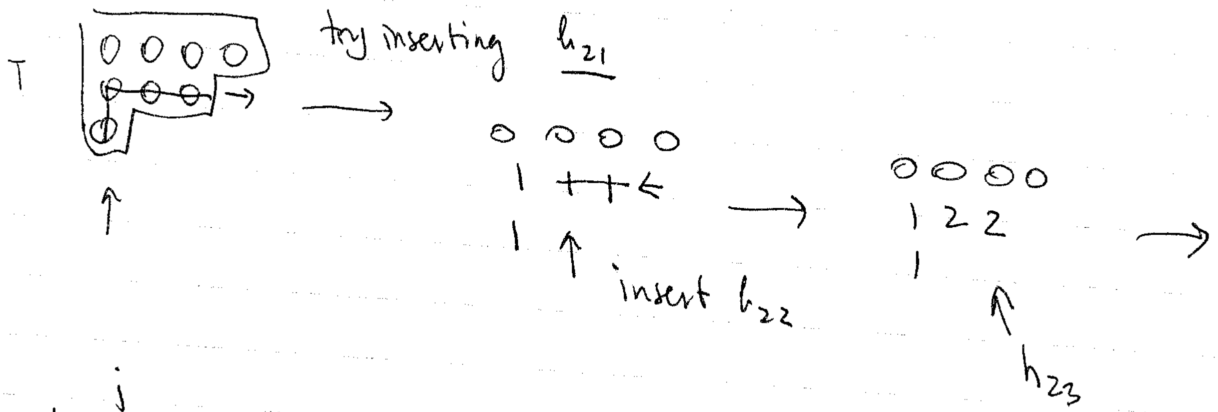
still equal.

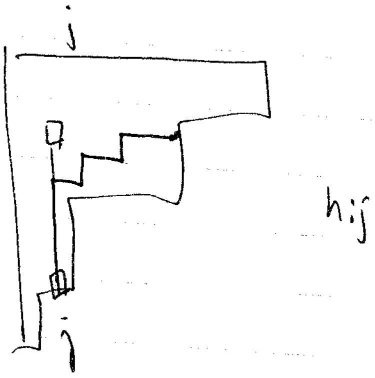
$\Rightarrow j' \leq j$.

Inverse procedure.

$F \rightsquigarrow (h_{i_1 j_1} h_{i_2 j_2} \cdots h_{i_r j_r})$

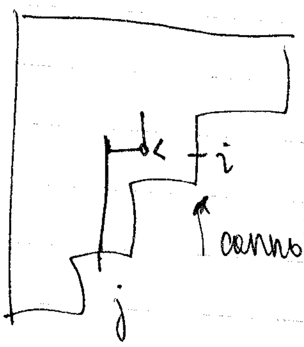
$F = \begin{array}{cccc} 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & \\ 0 & & & \end{array} \to h_{11} h_{11} h_{23} h_{22} h_{21}$



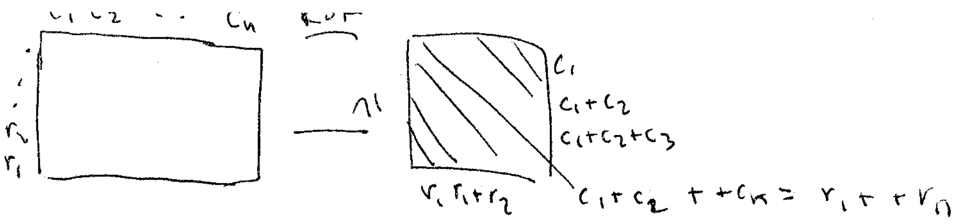


$\boxed{a}\boxed{b}$

if a = b go to right
o.w. go up } all horizontal steps corr to = entries

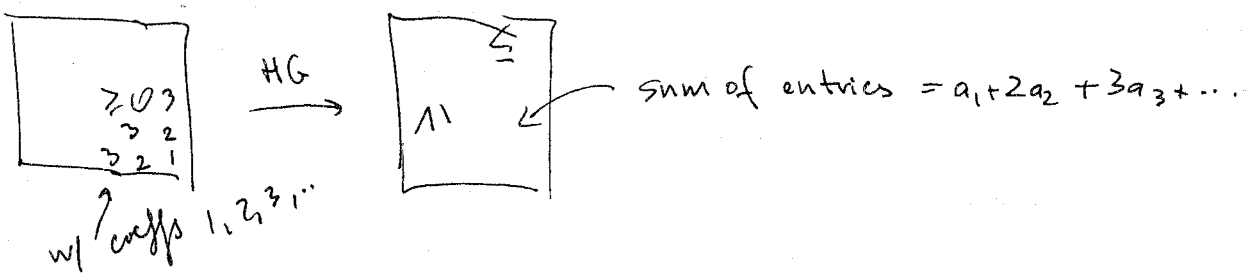


cannot arrive to i → this never happens.



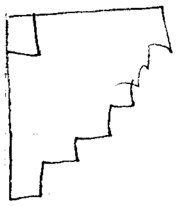
plane partition of square size.

So Hillman-Grassl corr. gives another bijection between matrices & plane partitions.



11/2/06.

③ (a) $S_{\lambda/1}(x_1,..x_n) = [y]\, s_\lambda(y, x_1,.., x_n) \underset{\text{since symm}}{=} [y]\, s_\lambda(x_1,.., x_n, y) = \sum_{\mu \nearrow \lambda} s_\mu(x_1 .. x_n).$



(b) same but take coeff of y^2

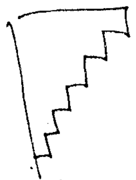
$s_{\lambda/2} = [y^2]\, s_\lambda(y, x_1 .. x_n) = [y^2]\, s_\lambda(x_1 .. x_n y) = \sum_{\substack{\mu = \lambda - \\ \text{horizontal} \\ \text{2-strip}}} s_\mu(x_1 .. x_n).$

$s_{\lambda/12} = \sum_{\nu - \lambda\text{-vertical 2 strip}} s_\nu$

apply $\underline{\omega}$ involution

$s_{\lambda/2} - s_{\lambda/12} = \sum_{\substack{\mu - \lambda\text{-horiz} \\ \text{domino}}} s_\mu - \sum_{\substack{\mu - \lambda\text{-vertical} \\ \text{domino}}} s_\mu$

(c) $= 0$ so no way to remove a horizontal or vertical domino $\Rightarrow \lambda =$



staircase shape.

④ RSK $A \to (P, Q)$
$A^T \to (Q, P)$
$A = A^T \to (P, P)$

④ $$F_n = \sum_{\lambda} \#SSYT(\lambda, n)\, q^{|\lambda|} = \sum_{\substack{A\text{-symm } n\times n \\ \text{matrix}}} q^{\Sigma a_{ij}} = \Big(\sum_{a_{11}} q^{a_{11}}\Big)\Big(\sum_{a_{22}} q^{a_{22}}\Big) \cdots \Big(\sum_{\substack{a_{12} \\ = a_{21}}} q$$

$$= \frac{1}{(1-q)^n} \frac{1}{(1-q^2)^{\binom{n}{2}}}$$

⑤ $$\#SSYT(\lambda, n) = s_\lambda(\underbrace{1,1..1}_{n}) = \det\left(\binom{n+\lambda_i - i + j}{\lambda_i - i + j}\right)_{i,j=1}^{n} = \prod_{i<j} \frac{\lambda_i - i - \lambda_j + j}{j - i}$$

↗ integer points in Gelfand-Tsetlin polytope. So if parts are large ↗ volume similar.

Volume was $\prod_{i<j} \frac{\lambda_i - \lambda_j}{j-i}$.

Ricky Liu

$\alpha_i = \lambda_i + n - i$. Want $\det\left(\binom{\alpha_i + j - 1}{n-1}\right)_{i,j=1}^{n}$

rows → $\binom{\alpha_i}{n-1}$ $\binom{\alpha_i+1}{n-1}$ .. $\binom{\alpha_i + n - 1}{n-1}$

column oper. → $\binom{\alpha_i}{n-1}$, $\binom{\alpha_i}{n-2}$, $\binom{\alpha_i+1}{n-2}$, .. $\binom{\alpha_i + n - 2}{n-2}$

$\binom{\alpha_i}{n-1}$, $\binom{\alpha_i}{n-2}$, $\binom{\alpha_i}{n-3}$ $\binom{\alpha_i+1}{n-3}$ ... $\binom{\alpha_i + n - 3}{n-3}$

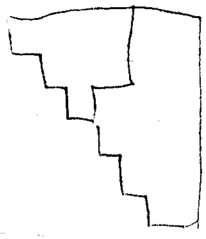
continue until we get:

$$\det\left(\binom{\alpha_i + j - 1}{n-1}\right)_{i,j=1}^{n} = \det\left(\binom{\alpha_i}{n-j}\right)_{i,j=1}^{n} = \prod \frac{1}{j(n-j)!} \det(P_{n-j}(\alpha_i)) =$$

$$\shortparallel \quad \frac{1}{(n-j)!}\left(\alpha_i(\alpha_i - 1) \cdots (\alpha_i - n + j + 1)\right)$$

$$= \prod_j \frac{1}{(n-j)!} \det(\alpha_i^{n-j})$$ ▨

⑥



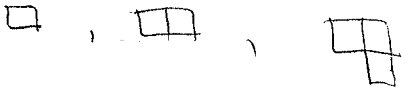
$$\sum s_{k,n} q^k = (1+q)(1+q^2)\cdots(1+q^n)$$

$+) \; a_i \le n$

expand it → terms correspond to shift partitions

so ~~product~~ problem is equivalent to fact that this poly has unimodal coefficients

Alexey: $n=2$: ε,

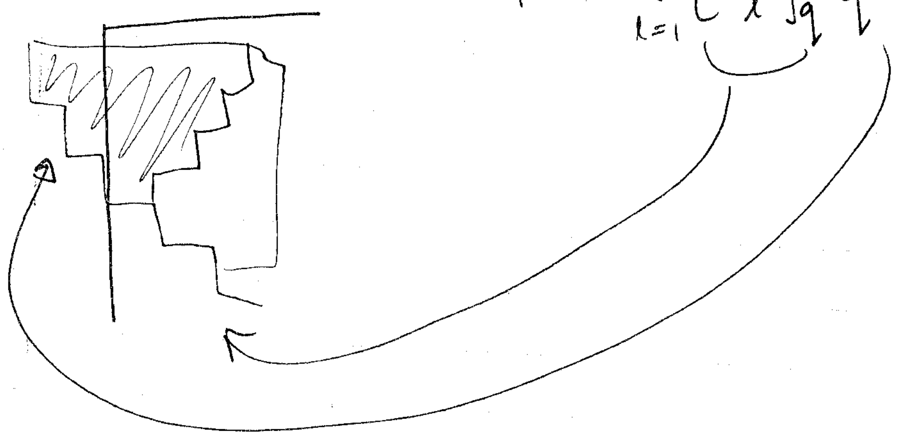


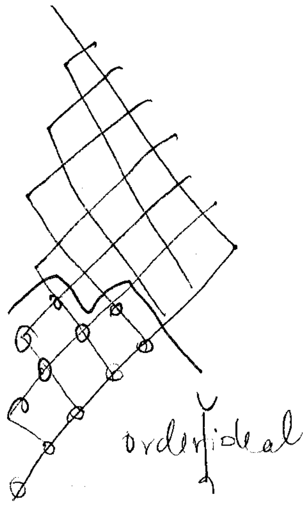
want to go from n to $n+1$

$n+1=3$

$n \to n+1$

$$\sum s_{k,n} q^k = (1+q)(1+q^2)\cdots(1+q^n) = \sum_{l=1}^{n} \underbrace{\begin{bmatrix} n+1 \\ l \end{bmatrix}_q}\, q^{\binom{l}{2}}$$

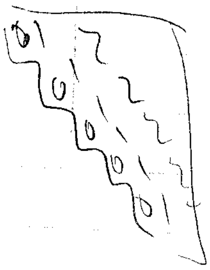




Need weight function $w: P \to \mathbb{R}_{>0}$

$$\sum_{x \in add(\lambda)} w(x) > \sum_{y \in remove(\lambda)} w(y)$$

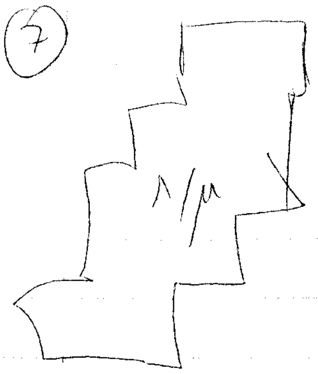
then can use this lemma & get them be unimodal.

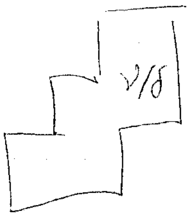


Weight fnct that works: $w(x) = (n - c(x) + 1)(n + c(x) - 2)$

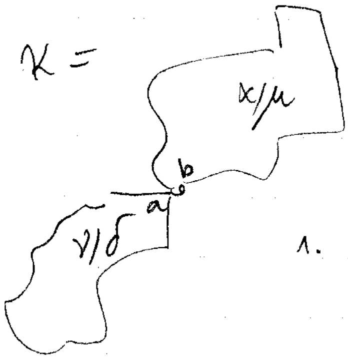
⑦

$s_{\lambda/\mu} \cdot s_{\nu/\delta} = s_\kappa = s_{\kappa'} + s_{\kappa''}$





$\kappa =$

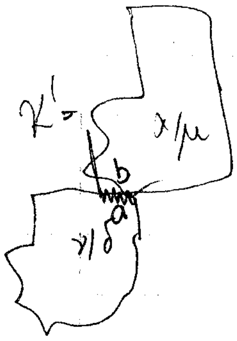


1. $a \leq b$

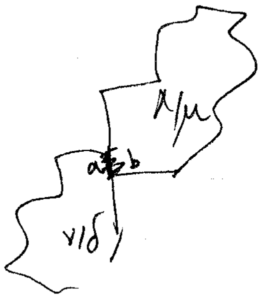
$\downarrow$

κ''

$\kappa' =$



$\kappa'' =$

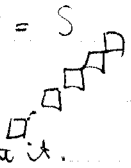


2. $a > b$

$\downarrow$

κ'

$s_1 = x_1 + x_2 + \dots$, $(s_1)^n = s$



$= \sum_{\kappa \text{ ribbons!}} s_\kappa .$ ☺

⑧ → to long, so we didn't write it.

⑨

$K_{\lambda,\mu} \leq K_{\lambda,\nu}$ $\mu \geq \nu$ in dominance order

$\mu = (\mu_1, \mu_2, \dots, \mu_n)$ $\nu = (\mu_1 \dots \mu_{i-1}\ \mu_i - 1,\ \mu_{i+1} \dots\ \mu_j + 1 \dots)$ $i < j$

go one step down
in dom. order
→ we know these
generate.

Kostka #'s don't change by perm weights: so let

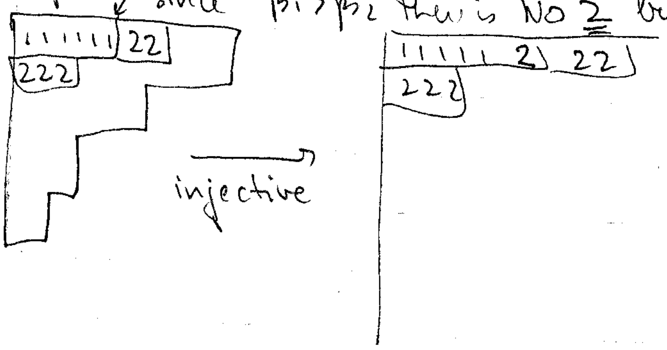
$\beta = (\mu_i, \mu_j, \mu_1 \dots) = (\beta_1, \beta_2, \dots)$ $\overset{\mu_i}{\overset{\shortparallel}{\beta_1}} > \overset{\mu_j}{\overset{\shortparallel}{\beta_2}}.$

$\gamma = (\beta_1 - 1, \beta_2 + 1, \beta_3, \dots)$

$K_{\lambda\beta} \leq K_{\lambda\gamma}$

any composition s.t. $\beta_1 > \beta_2$

β_1 since $\beta_1 > \beta_2$ there is NO 2 below last 1

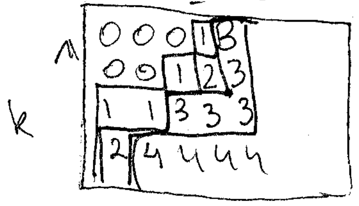


← subset of tableau of weight γ & shape λ.

④ specialize at $x_i = q$.

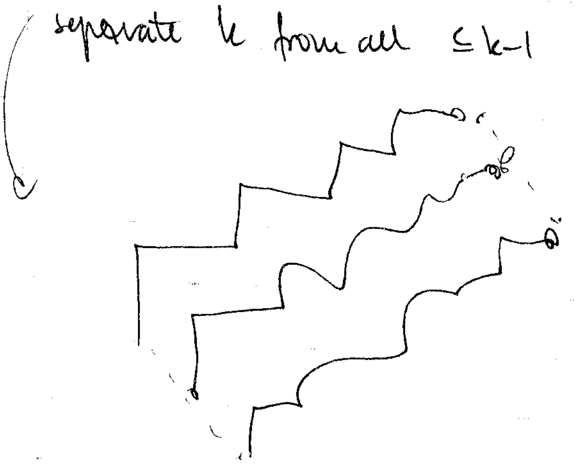
$$\sum_{SSYT} x^{\text{weight}(T)} = \prod \frac{1}{1-x_i} \prod_{i<j} \frac{1}{1-x_i x_j}$$. by RSK. (symm version)

RPP of rectangular shape $k \times l$



we can move them away so they have no common pts.

separate k from all $\leq k-1$ by path. get nonintersecting paths.



$R(k,l,m) = \#\,RPP\,(k \times l)$ with entries $\leq$

Lindström lemma $\Rightarrow$

$$R(k,l,m) = \det\left(\binom{k+l}{k+i-j}\right)_{i,j=1}^{m} = s_{k \times m}(\underbrace{1,\ldots,1}_{k+l})$$

$$\uparrow$$

$$e_{k+i-j}(\overbrace{1\ldots1}^{k+l})$$

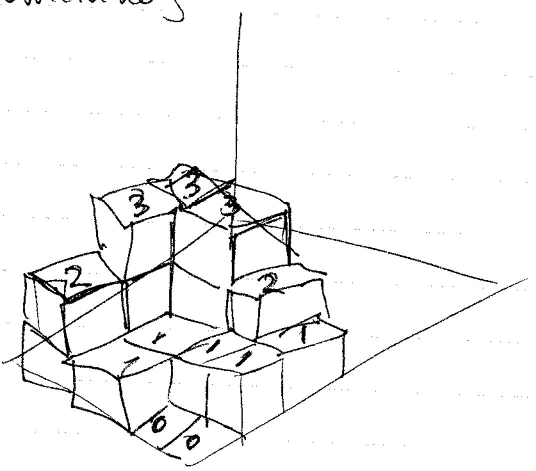
also $R(k,l,m) = s_{k \times l}(\underbrace{1\ldots1}_{k+m})$

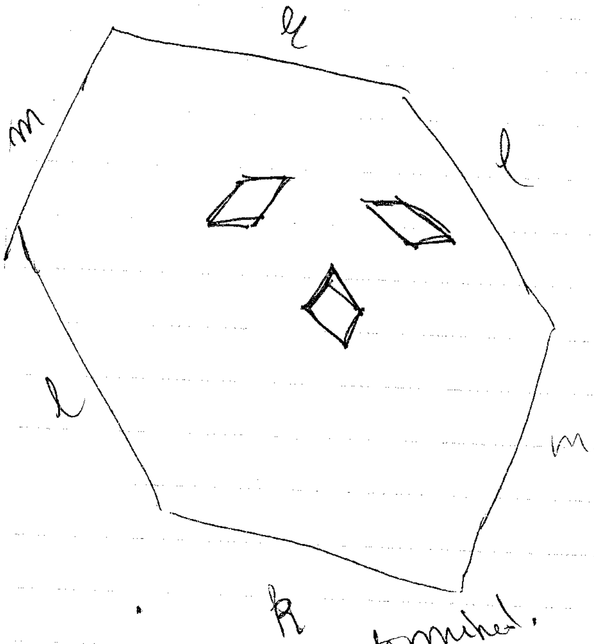
Theorem. $R(k,l,m) = \prod_{a=1}^{k} \prod_{b=1}^{l} \prod_{c=1}^{m} \frac{a+b+c-1}{a+b+c-2}$.

See this symmetry immediately geometrically

Look at PP:

$$\boxed{\begin{matrix} 3 & 3 & 2 & 1 \\ 3 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 \end{matrix}}$$





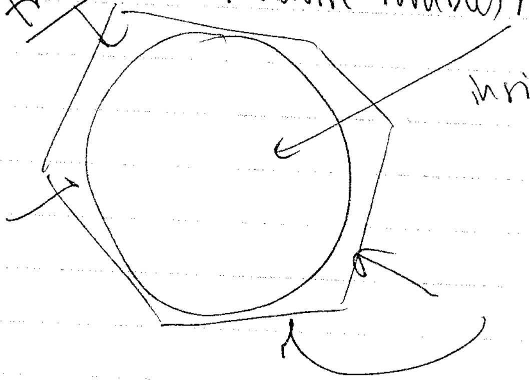
rombus tiling of ~~rombus~~ hexagon

3 different rombuses

counts # to subdivide hexagon into rombuses.

frozen → predetermined.

random rombus tiling inside



$\left[\begin{smallmatrix} n\\ k\end{smallmatrix}\right]_q = \sum\limits_{\lambda\subseteq k\times(n-k)} q^{|\lambda|}$; 3-dim analog

11/07/06

Theorem. $\sum\limits_{\text{plane partitions } T\subseteq k\times l\times m} q^{|T|} = \prod\limits_{a=1}^{k}\prod\limits_{b=1}^{l}\prod\limits_{c=1}^{m}\frac{[a+b+c-1]_q}{[a+b+c-2]_q}$

$T = k\ \overset{l}{\boxed{\quad \le m\quad}}$

for higher dimensions - no nice formula

$|T| = \sum T_{ij}$

Proof based on hook-content formula from PSET4.

$$s_\lambda(1,q,q^2,\dots,q^{n-1}) = q^{m(\lambda)}\prod_{x\in\lambda}\frac{[n+c(x)]_q}{[h(x)]_q}$$

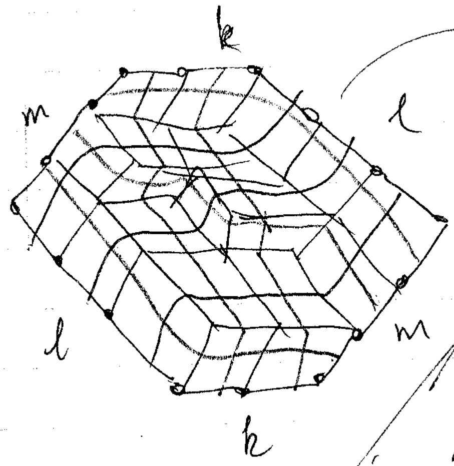
$$k\ \underset{l}{\boxed{\text{SSYT's}_{\le n}}} \xrightarrow{\sim} k\ \underset{l}{\boxed{\text{RPP}_{\le n-k}}}$$

$$T \rightsquigarrow \tilde{T}$$

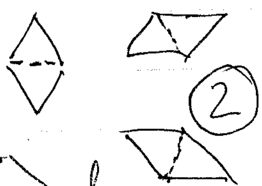
$$\sum T_{ij} \longrightarrow l\binom{k}{2} + \sum \tilde{T}_{ij}$$

$$\sum_{T\subseteq k\times l\times m} q^{T} = q^{-m(k\times l)}\, s_{k\times l}(1,q,\dots,q^{m+k-1}) \quad \boxtimes$$

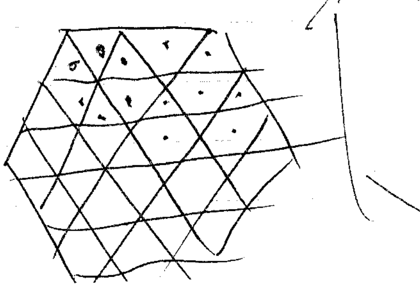
①

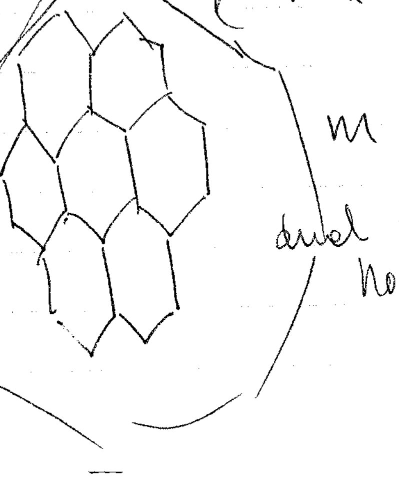


how should we describe these in terms of honeycomb graphs?



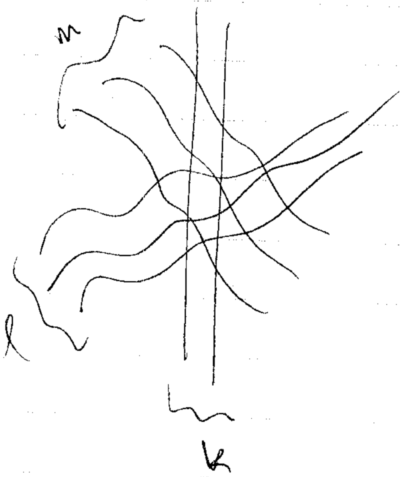
② perfect matchings on honeycomb ☺



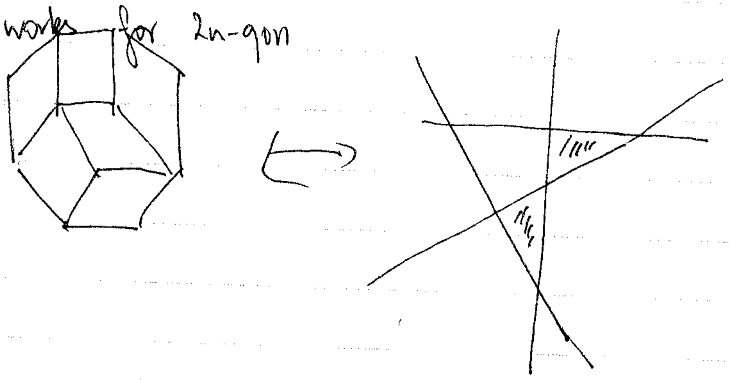


dual graph looks like honeycomb.

③ pseudoline arrangements: collection of curves such that (1) no triple intersection is allowed, and (2) any 2 curves intersect in at most 1 point



Same works for 2n-gon



n lines s.t. any pair of lines intersect eachother.

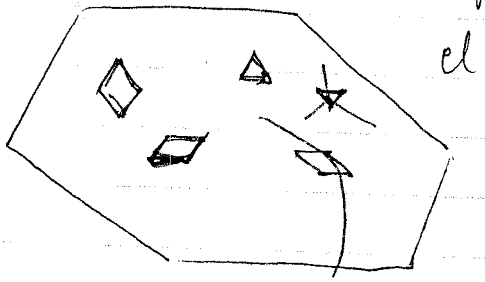
pairwise intersecting

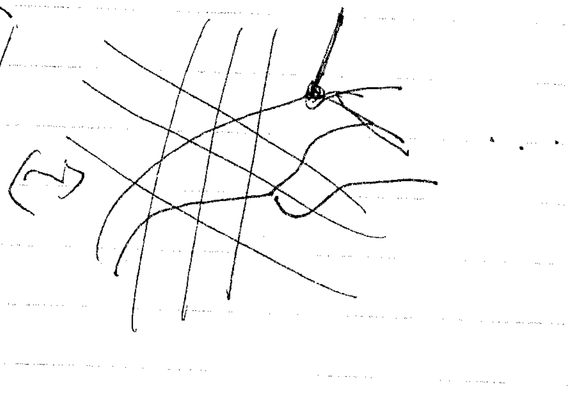
Exercise (a) Let A be a pseudoline arr w/ n lines. It subdivides $\mathbb{R}^2$ into regions. Show that there are $\geq n-2$ triangular regions.

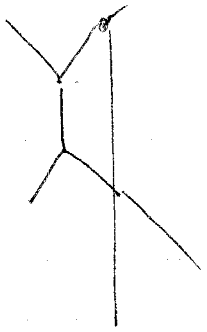
(Btw, total # of regions $n(n+1)$)

(b) find a pseudoline arr which is not line arrangement.

asymptotics; connected to electromagnetics

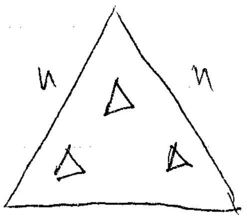






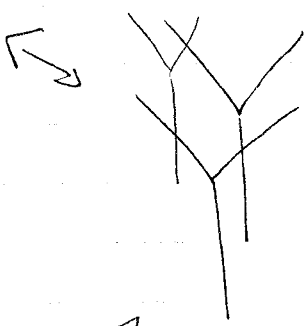
Feynman diagrams.

Y ← have trivalent vertices.



need at least n Δ to subdivide this

n "holes" Δ



superimposition of n letters Y

↗ combinatorial types of drawing

no explicit formulas

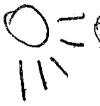
subdivide
product of simplices

$\Delta^n \times \Delta^m \to$ this is how this came up.

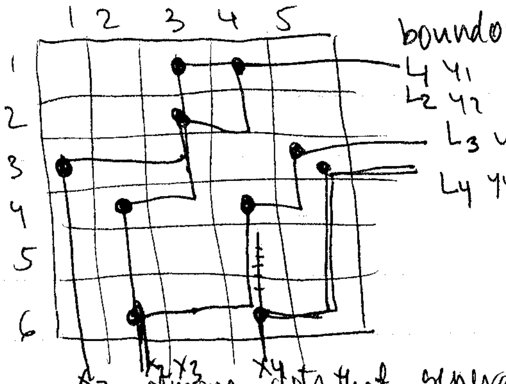
baricentric coordinates are more convenient to work with.

<u>RSK</u>: $A \to (P,Q)$ $A^T \to (Q,P)$ symmetry is not very clear from defn.

one way to explain it is thru Fomin's growth diagrams

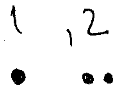


Other way: <u>Viennot construction</u>

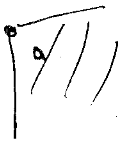


boundary shadow

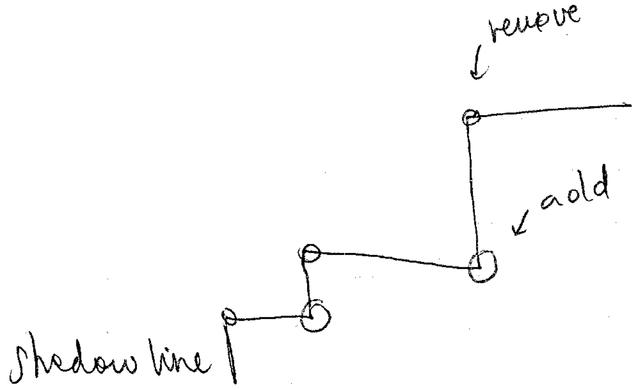
instead



shadow of dot

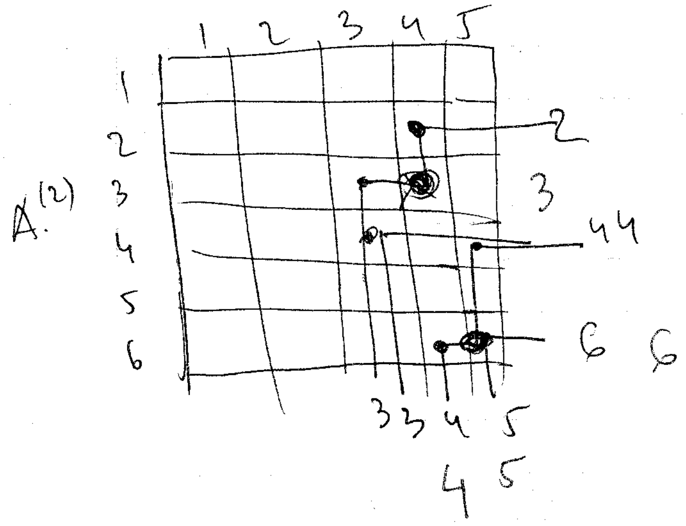


remove dots that generate shadow line → look again



$$P = \begin{array}{|cccc|} 1 & 2 & 2 & 4 \\ 3 & 3 & 4 & 5 \\ 4 & 5 & & \end{array}$$

$$Q = \begin{array}{|cccc|} 1 & 1 & 3 & 3 \\ 2 & 4 & 4 & 6 \\ 3 & 6 & & \end{array}$$



Theorem (Viennot) The map $A \to (P, Q)$ described thru the shadow line technique is the RSK-correspondence.

$$A' = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

gen. perm.

$$W = \begin{pmatrix} 1 & 1 & 2 & 3 & 3 & 3 & 4 & 4 & 6 & 6 \\ 3 & 4 & 3 & 1 & 5 & 5 & 2 & 4 & 2 & 4 \end{pmatrix}$$

<u>RSK</u>

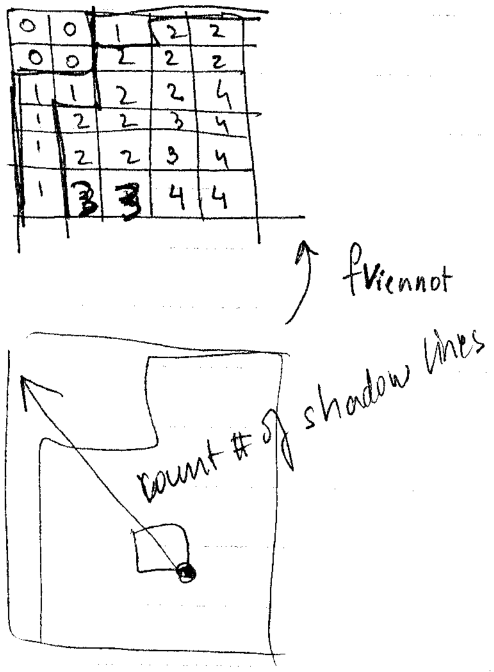
P: $3 \quad 34 \quad 33_{\swarrow 4} \quad 13_{\swarrow 3} \quad 135 \quad 1355 \quad 1255_{\swarrow 3}$

just first row is considered

$\hookrightarrow 1245_{\swarrow 5} \to 1225_{\swarrow 4} \to \underline{1224}$

Q: $1 \quad 11 \quad 11 \quad 11 \quad 113 \quad \boxed{1133}$

nonintersecting paths ⟷ plane partitions



0	0	1	2	2
0	0	2	2	2
1	1	2	2	4
1	2	2	3	4
1	2	2	3	4
1	3	3	4	4

Assume we have $n \times n$ matrices & $n \times n$ RPPs.

RSK

f_{HG} Hillman Grassl

f_{Viennot}

pair of tableaux → Gelfand-Tsetlin patte

→ put together -matrix.

columns of shadow lines $x_1, x_2 \ldots$

in Example $x_1 = 1$, $x_2 = x_3 = 2$, $x_4 = 4$

rows $y_1 = 1$, $y_2 = 1$, $y_3 = y_4 = 3$

total # of shadow lines is $\ell(sh(P^\dagger)) = \ell(sh(Q^\dagger))$

$P = \begin{array}{|cccc|} \hline 1 & 2 & 2 & 4 \\ & & & \\ \end{array}$ $\quad Q = \begin{array}{|cccc|} \hline 1 & 1 & 3 & 3 \\ & & & \\ \end{array}$

move a horizontal line from above & look at what shadow lines it interest & record column in which it intersects

P $\boxed{3}$ 3 4 3 3 1 3 ...

if dots in same row what is to the left is a bit higher.

Q .... [illegible] ?

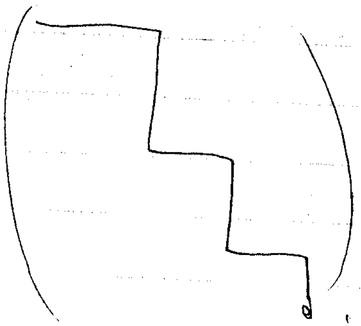
other rows also decorr. $A^{(2)}$ → corr to <u>bumped</u> entries

So by induction see $A^{(2)}$ corr to 2nd row of tableaux etc.

One more way to explain the symmetry of RSK.

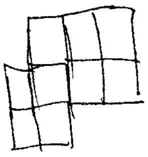
first row of P, Q has length = total # of shadow lines

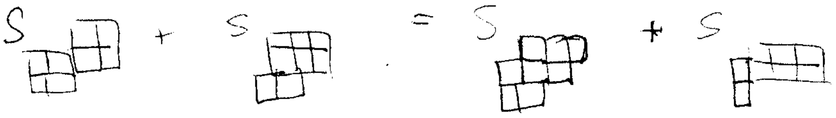
how to find it from A?



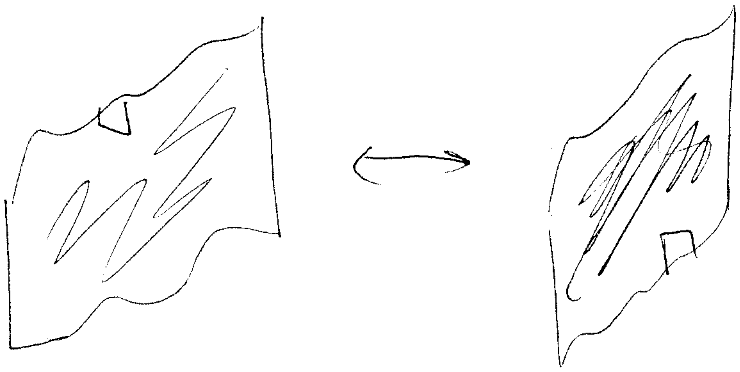
just find path s.t. to what the elts in it add up is maximal over all paths.

λ/μ **Claim.** $\sum_{\nu \searrow \mu} s_{\lambda/\nu} = \sum_{\gamma \nearrow \lambda} s_{\gamma/\mu}$ 11/07/06

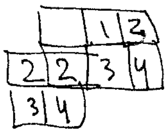




can be proved using symmetry of Schur fnct



same collection of numbers as in other tableau



$$\begin{array}{cccc} & 1 & \sqcup & 3 \\ 2 & 2 & 3 & 4 \\ 3 & 4 & & \end{array} \to \begin{array}{cccc} & 1 & 3 & 3 \\ 2 & 2 & \sqcup & 4 \\ 3 & 4 & & \end{array} \to \cdots$$

jeu de taquin