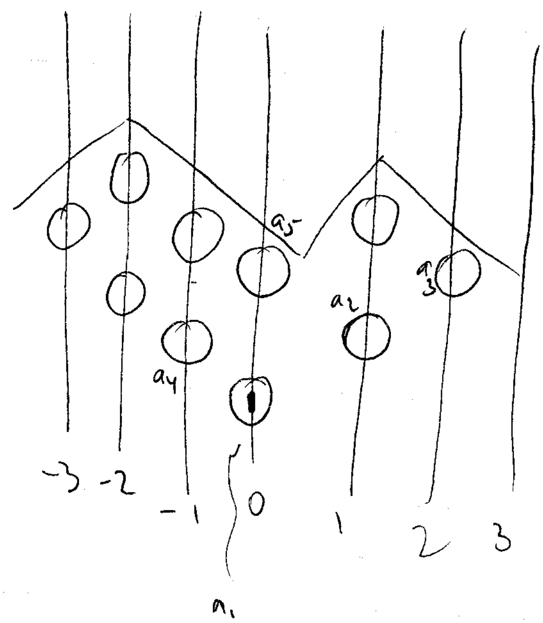
$\alpha = (\ldots \overset{i}{a}\ \overset{i+1}{a+1}\ \overset{i+2}{a} \ldots)$. By prop 2)

$$s_i s_{i+1} s_i v_\alpha = -v_\alpha$$

$$s_{i+1} s_i s_{i+1} v_\alpha = v_\alpha$$

so $v_\alpha = -v_\alpha$ ↯ since it is a basis vector.

follows that a_i's are integers.



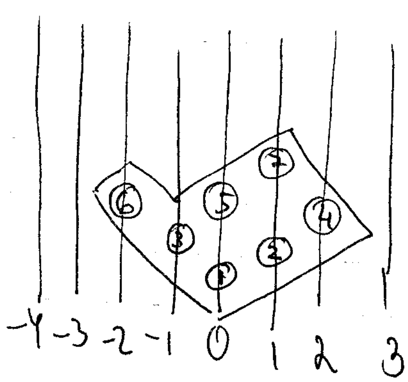
<u>09/28/06</u>

$\mathrm{Spec}(n) \subseteq \mathrm{Cont}(n)$

content vectors $(a_1, \ldots, a_m)$

<u>Ex.</u> $\alpha = (\overset{1}{0}, \overset{2}{1}, \overset{3}{-1}, \overset{4}{2}, \overset{5}{0}, \overset{6}{-2}, \overset{7}{1})$

strings labeled by all integers



Young tableau (filled w/ numbers)

⤳ rotate it & get:
and reflection

1	2	4
3	5	7
6		

one cannot pass through the other

can see by induction that the shape we get is Young diagram

allowed places $\leftrightarrow$ corners of diagram where we can add & still get a Y.D.

Def. content of a box $x=(i,j)$ in Y.D. is $c(x) = j-i$

$$\begin{array}{ccccc} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & -1 & 0 & 1 \\ & & & -1 & \end{array}$$

Tableau $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 7 \\ \hline 6 \\ \cline{1-1}\end{array}$ $(0, 1, -1, 2, 0, -2, 1)$

vector of contents of T

Claim. Cont(n) is in bijection with set of content vectors of tableaux. ✓

\# 0's — size of 0'th diag

\# 1's — 1'st —

\# −1's — " —

get shape of corresp. tableau!

Admissible transpositions : $(0, 1, -1, 2, 0, -2, 1)$

corr. to switching 4, 5

correspond to switches of boxes labeled i and $i+1$ provided they are not in adjacent diags (~~their~~ contents of their boxes differ >1)

We never change $sh(T)$ by this. Basically we can switch if result is a valid tableau.

Lemma. Any two SYT of same shape can be obtained from each other by these admissible transpositions.

Proof. Try to transform any T into the form

$T_0 = \begin{array}{l} 1\ 2\ 3 \cdots k \\ k+1 \cdots \ell \\ \ell+1 \cdots \end{array}$

Given T, find first box where T differs from T_0.

$T = \begin{array}{l} 1\ 2\ 3\ 4\ 7 \cdots \\ \quad\quad 6 \end{array}$?? 6 can never be adj to 7 $\rightarrow$ switch 6 & 7

switch, then find 5 $\rightarrow$ do same.

and so on... proceed by induction.

~~Thm~~ $\underline{\mathrm{Spec}(n), \sim}$ equiv relation

$\mathrm{Cont}(n), \approx$: $\alpha \approx \beta$ if α can be obtained from β by admissible transpositions

So far we proved (1) $\mathrm{Spec}(n) \subseteq \mathrm{Cont}(n)$

(2) $\forall \alpha \in \mathrm{Spec}(n)$, $\beta \in \mathrm{Cont}(n)$ s.t. $\beta \approx \alpha$ then

$\beta \in \mathrm{Spec}(n)$ and $\alpha \sim \beta$.

These two facts $\Rightarrow$ $\mathrm{Spec}(n)/\sim \;\subseteq\; \mathrm{Cont}(n)/\approx$

We know $|\mathrm{Spec}(n)/\sim| = \#$ irreps $= \#$ conj. classes $= |S_n^\vee| = \#$ partitions of n

2 vectors are $\sim$ if they are in same irrep

$|\mathrm{Cont}(n)/\approx| = \#$ Ydiags. $= \#$ partitions of n

$|\mathrm{Spec}(n)/\sim| = |\mathrm{Cont}(n)/\approx|$

$\Rightarrow$ $\mathrm{Spec}(n) = \mathrm{Cont}(n)$ & $\sim \, = \, \approx$.

<u>Theorem.</u> $\mathrm{Cont}(n) = \mathrm{Spec}(n)$, $\sim \, = \, \approx$.

$\sim$ equivalence classes

<u>or.</u> $S_n^\vee \rightsquigarrow$ Young diagrams

GT-basis of V_λ $\rightsquigarrow$ SYT of shape λ

<u>Cor.</u> $\alpha \sim \beta \iff \alpha, \beta$ equal as multisets

(1), (2) $\Rightarrow$ each $\sim$-equiv. class is a union of some $\approx$-equiv classes.

On the other hand $|\mathrm{Spec}(n)/\sim| = |\mathrm{Cont}(n)/\approx|$. $\Rightarrow$

$\mathrm{Spec}(n)/\sim \; = \mathrm{Cont}(n)/\approx$

<u>Branching rule.</u> $\lambda \in S_n^\vee$, $\mu \in S_{n-1}^\vee$ ~~then~~

$\mathrm{Res}^{S_n}_{S_{n-1}} V_\lambda$ contains V_μ iff μ is obtained from λ by removing a corner.

~~Th~~ <u>Proof.</u> Suppose $v_\alpha \in$ GT-basis of V_λ, then if take $\mathrm{Res}^{S_n}_{S_{n-1}} V_\lambda$, then just remove last entry of α to get $\bar{\alpha}$, $v_{\bar{\alpha}} \in$ GT-basis w.r.t. S_{n-1}

in terms of tableau, dropping last entry of α $\rightarrow$ deleting n from T

$v_\alpha \in V_\mu$ where μ is obtained from λ by removing a corner.

$f^\lambda = \# SYT(\lambda)$

<u>Hook-length formula</u>: $f^\lambda = \dfrac{n!}{\prod_{x\in\lambda} h(x)}$

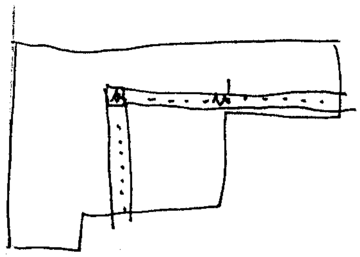
$H(\lambda) = \prod_{x\in\lambda} h(x)$.

$f^\lambda = \sum_{\mu \nearrow \lambda} f^\mu$. Can try proving by induction.

Need some recurrence for $\dfrac{n!}{H(\lambda)}$

"Hook walk proof" due to Green, Nijenhuis, Wilf.

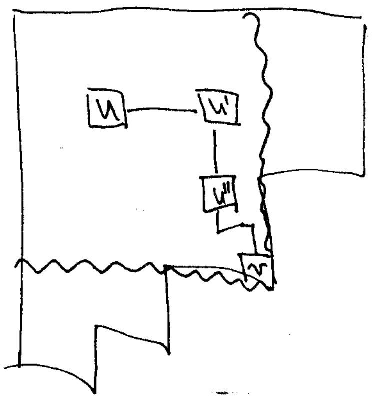
Zeilberger: "most beautiful proof in mathematics"



probability of going from u to u':

$$\Pr(u,u') = \frac{1}{h(u)-1}$$

So we just jump around while we get to some corner.

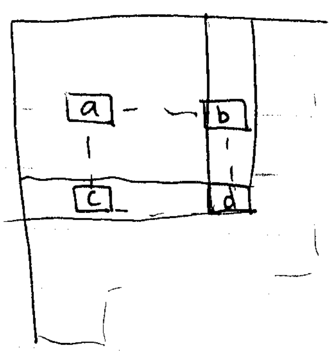


$P(u,v)$ = prob that walk starts at u, ends at v (any number of jumps)

particular box / particular corner

$$P(u,v) = \sum_{\text{paths from } u \text{ to } v} \frac{1}{(h(u)-1)\underbrace{(h(u')-1)}_{\text{weights of vertices } wt(u')}(h(u'')-1)\cdots}$$

Observation. If you fix corner v everything happen in rectangle



$h(a)+h(d) = h(b)+h(c)$

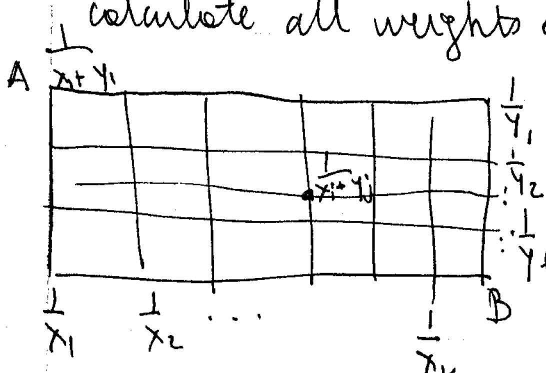
Now, if d is a corner:

$h(a)-1 + \cancel{h(d)-1}^{\,1} = h(b)-1 + h(c)-1$

$wt(a) := \dfrac{1}{h(a)-1}$

$\Rightarrow$ if $wt(b) = \frac{1}{x}$, $wt(c) = \frac{1}{y}$ $\Rightarrow$ $wt(a) = \dfrac{1}{x+y}$

If we know all weight of boxed above & to left of d $\to$ can calculate all weights of all boxes in rectangle by above formula

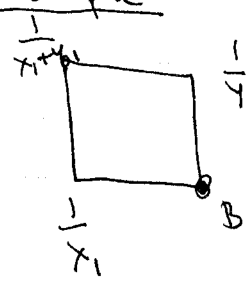


Lemma 1.
$$\sum_{\text{lattice paths } P:A\to B} wt(P) = \frac{1}{x_1x_2\cdots x_k\,y_1\cdots y_\ell}$$

$\binom{k+1}{\ell}$

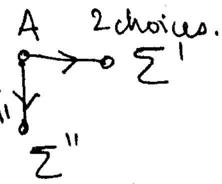
product of wt's

Example $k=l=1$



$$\frac{1}{x_1+y_1}\cdot\frac{1}{y_1}+\frac{1}{x_1+y_1}\cdot\frac{1}{x_1}=\frac{1}{x_1 y_1}$$

Proof. By induction.



$$\Sigma=(\Sigma'+\Sigma'')\frac{1}{x_1+y_1}$$

$$\Sigma'=\frac{1}{x_2x_3\cdots x_k y_1\cdots y_l}$$

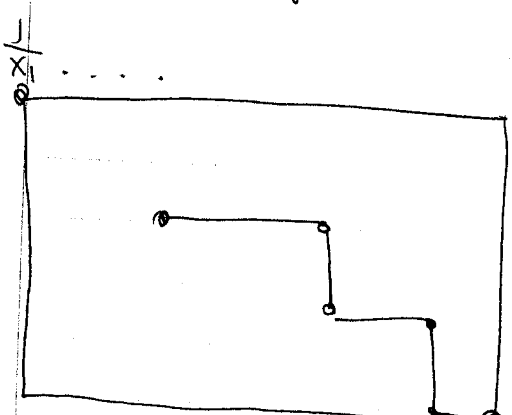
$$\Sigma''=\frac{1}{x_1\cdots x_k y_2\cdots y_l}$$

$$\Big\}\ \Sigma=\frac{1}{x_1+y_1}\left(\Sigma'+\Sigma''\right)$$

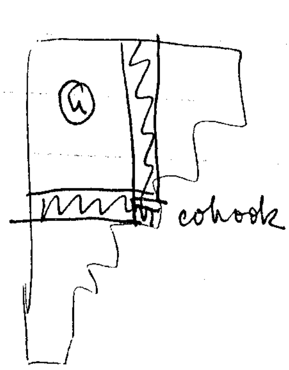
$$\Rightarrow\ \Sigma=\frac{1}{x_1\cdots x_k y_1\cdots y_l}$$

Lemma 2. $$\sum_{\substack{\text{hook walk paths } P\\ \text{(in same rectangle)}}} wt(P)=\left(1+\frac{1}{x_1}\right)\cdots\left(1+\frac{1}{x_k}\right)\left(1+\frac{1}{y_1}\right)\cdots\left(1+\frac{1}{y_k}\right)$$

because, open up parenthesis ...



Back to hooks: Fix corner $v\in\lambda$



$$\sum_{\substack{u\\ \text{in rectangle}}} P(u,v)=\prod_{t\in \text{cohook}(v)}\left(1+\frac{1}{h(t)-1}\right)=\prod_{t\in \text{coh}(v)}\frac{h(t)}{h(t)-1}=\frac{H(\lambda)}{H(\lambda/v)}$$

hook length polynomial

$$\sum_{v \text{ corners}} P(u,v) = 1 \quad \swarrow \text{fixed } u$$

$$\sum_u \Big(\sum_{v \text{ corners}} P(u,v) \Big) = n \qquad \text{as there are } n \text{ choices for } u$$

$$= \sum_{v \text{ corners}} \frac{H(\lambda)}{H(\lambda/v)}$$

$$\Rightarrow \frac{n!}{H(\lambda)} = \sum_v \frac{(n-1)!}{H(\lambda/v)}$$

, which is exactly the recurrence relation we need.

10/03/06

Schur functions

Combinatorial definition, based on semi-standard Young tableaux (SSYT)

$$T = \lambda \quad \begin{array}{lllllll} & & \leq & & & & \\ 1 & 1 & 1 & 2 & 3 & 3 & 5 \\ 2 & 2 & 3 & 5 & & & \\ 5 & & & & & & \end{array}$$

$sh(T) = \lambda$

$\text{weight}(T) = \beta = (\beta_1, \beta_2 \ldots)$ $\qquad \beta_i = \#i\text{'s in } T$

Kostka number $K_{\lambda\beta}$ = # tableaux (SSYT) of shape λ and weight β

Schur functions $s_\lambda = \sum_{T - SSYT(\lambda)} x^{\text{weight}(T)}$ $= x_1^{\beta_1} x_2^{\beta_2} \ldots$

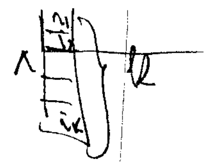
$$= \sum_\beta K_{\lambda,\beta} \, x^\beta$$

Example $s_{(1)} = x_1 + x_2 + x_3 + \ldots$ $\leftarrow$ not poly, since so many variables.

Schur polynomials $s_\lambda(x_1, \ldots, x_n) = \sum_{T - SSYT(\lambda)} x^{\text{weight}(T)}$

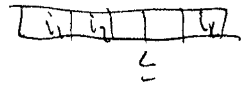
$\nearrow$ has entries $\leq n$.

It will be more convenient to use infinitely many variables



Ex. (2) $s_{(1^k)} = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ ← k-th elementary symmetric functs

Ex (3)



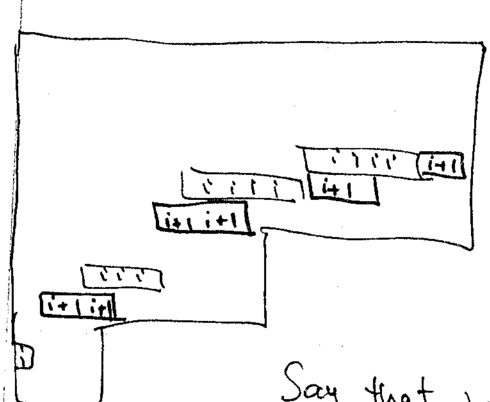
$s_{(k)} = \sum_{i_1 \le \cdots \le i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ ← k-th complete homogeneous symm f

Lemma. s_λ is a symmetric funct (w.r.t. all possible perm of variables x_i) that is, $K_{\lambda\beta}$ are the same for all possible permutation of β

Proof. Enough to prove that $s_\lambda(\ldots x_i, x_{i+1} \ldots) = s_\lambda(\ldots x_{i+1}, x_i \ldots)$, equivalently $K_{\lambda,(\beta_1 \ldots \beta_i \beta_{i+1} \ldots)} = K_{\lambda,(\beta_1 \ldots \beta_{i+1} \beta_i \ldots)}$.

Want bijection: $T \mapsto \tilde{T}$ s.t. $sh(T) = sh(\tilde{T})$, weight $(T) = (\ldots \beta_i \, \beta_{i+1} \ldots)$, weight $(\tilde{T}) = (\ldots \beta_{i+1} \, \beta_i \ldots)$

$T =$

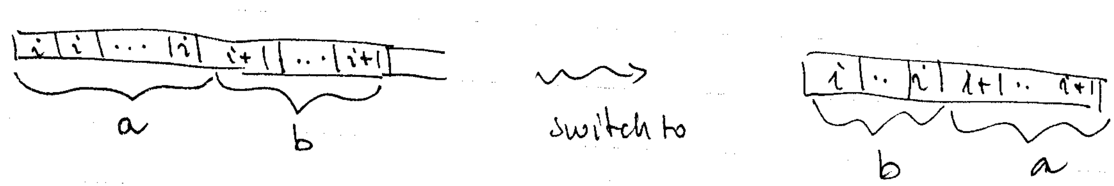


boxes of T filled w/ i's form a **horizontal strip**.

Say that pairs of boxes like $\begin{array}{|c|}\hline i \\ \hline i+1 \\ \hline\end{array}$ are <u>blocked</u>.

blocked i's = # blocked $(i+1)$'s

Unblocked i's and $(i+1)$'s form several components like



This is the needed map $T \to \tilde{T}$. Inverse is easy → just repeat procedure.
→ so it is a BIJECTION ☺

<u>Exercise.</u> Let $\tilde{s}_i : T \to \tilde{T}$. Clearly $\tilde{s}_i^{\,2} = 1$, $\tilde{s}_i \cdot \tilde{s}_j = \tilde{s}_j \tilde{s}_i$ $|i-j|>1$

In general: $\tilde{s}_{i+1} \tilde{s}_i \tilde{s}_{i+1} \neq \tilde{s}_i \tilde{s}_{i+1} \tilde{s}_i$

they don't <u>extend</u> to action of symmetric group on tableaux in obvious way

Problem 9*: Find operations on T that transform weight as needed (here) an[d]
satisfy Coxeter relations.

Cauchy identity

$$\sum_{\lambda} s_\lambda(x_1, x_2, \dots)\, s_\lambda(y_1, y_2, \dots) = \prod_{i,j\geq 1} \frac{1}{1 - x_i y_j}$$

λ – all partitions of all possible sizes, including $\emptyset$

$s_\emptyset(x_1 \dots) = 1.$

Robinson – Schensted – Knuth (RSK) correspondence

Forbenious identity: $\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$ (up-down ops / repr. theory)

We would like a bijective proof.

Schensted correspondence: $w \in S_n \longleftrightarrow \{(P, Q):$ two SYT of same shape $\lambda \vdash$

P - insertion tableau

Q - recording tableau

Take $w = 3\,7\,1\,2\,5\,4\,6$

P	$\emptyset$	$\begin{array}{c}3\end{array}$	$\begin{array}{cc}3&7\end{array}$	$\begin{array}{cc}1&7\\3\end{array}$	$\begin{array}{cc}1&2\\3&7\end{array}$	$\begin{array}{ccc}1&2&5\\3&7\end{array}$	$\begin{array}{ccc}1&2&4\\3&5\\7\end{array}$	$\begin{array}{cccc}1&2&4&6\\3&5\\7\end{array}$
Q	$\emptyset$	$\begin{array}{c}1\end{array}$	$\begin{array}{cc}1&2\end{array}$	$\begin{array}{cc}1&2\\3\end{array}$	$\begin{array}{cc}1&2\\3&4\end{array}$	$\begin{array}{ccc}1&2&5\\3&4\end{array}$	$\begin{array}{ccc}1&2&5\\3&4\\6\end{array}$	$\begin{array}{ccc}1&2&5\\3&4\\6\end{array}$

RSK

works w/ SSYT and generalized permutations

$A = (a_{ij})_{i,j\geq 1}$ $\quad a_{ij} \in \mathbb{Z}_{\geq 0}$ $\quad \infty \times \infty$ matrix w/ finitely many nonzero entr[ies]

generalized permutation: $w_A = \begin{pmatrix} i_1 \dots i_m \\ j_1 \dots j_m \end{pmatrix}$ s.t. (1) $i_1 \leq \dots \leq i_m$

(2) $i_k = i_{k+1} \Rightarrow j_k \leq j_{k+1}$ (3) for each pair (i,j) there are exac[tly]

a_{ij} columns $\binom{i}{j}$ in w_A.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad w_A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 2 & 2 & 1 & 2 \end{pmatrix}$$

RSK maps w_A to (P, Q), two SSYT, P filled w/ j's,
Q — " — i's

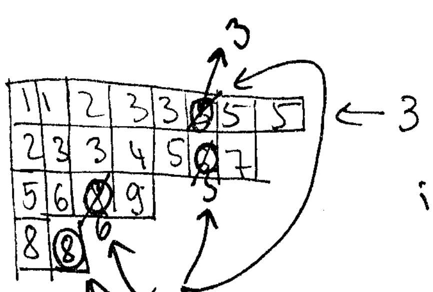
<u>Row insertion</u> P - SSYT, $k \in \mathbb{Z}_{\geq 1}$, $P \leftarrow k$ result of insertion

Denote by P_i = i-th row of P : (0) set $k_1 = k$

(1) if $k_1 \geq$ all entries in P_1 then just a new box $\boxed{k_1}$ at end of P_1.

(2) otherwise, let k_2 = first entry in $P_1 > k_1$ then k_1 "bumps" k_2 into second row P_2. Need to replace k_2 w/ k_1 and try insert k into P_2, and continue until nothing bumps but is added at end of some row.

Example



insertion path: $I(P \leftarrow k) = \{(1,6), (2,6), ($
$(4,2)\}$

different from starting tableau

RSK $w_A = \begin{pmatrix} 1 1 1 & 2 2 & 3 3 \\ 1 3 3 & 2 2 & 1 2 \end{pmatrix}$

P	$\emptyset$	$\boxed{1}$	1 3	1 3 3	1 2 3 / 3	1 2 2 / 3 3	1 1 2 / 2 3 / 3	1 1 / 2 : / 3
Q	$\emptyset$	$\boxed{1}$	1 1	1 1 1	1 1 1 / 2	1 1 1 / 2 2	1 1 1 / 2 2 / 3	1 : / 2 / 3

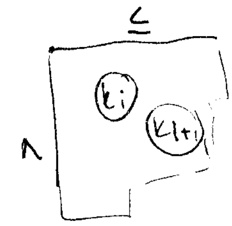
Insertion path $I(P \leftarrow k) :=$ positions of circled entries

<u>Lemma</u> (1) $I(P \leftarrow k)$ has form $\{(1, a_1), (2, a_2), .., (r, a_r)\}$, $a_1 \geq a_2 \geq .. \geq a_r$

(2) $k \leq l$ $\quad I(P \leftarrow k) = \{(1, a_1), .., (r, a_r)\}$, $I((P \leftarrow k) \leftarrow l) = \{(1, b_1), .., (s,$

Then $s \leq r$ and $b_1 > a_1$, $b_2 > a_2$, .., $b_s > a_s$. In particular b_s is always to the <u>right</u> of a_r.

Proof. (1) Need to show that we cannot have



k_{i+1} = first entry in P_{i+1} s.t. $k_{i+1} > k_i$. But the entry below k_i is already >

(2) Let $k_1, k_2, \dots$ be the bumped entries $P \leftarrow k$ and $l_1, l_2, \dots$ for $(P \leftarrow k) \leftarrow l$

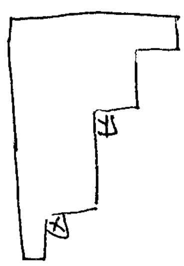
$k_1 < k_2 < \dots$

$l_1 < l_2 < \dots$ in P l_2 is located strictly to the right of k_2.

So $a_1 < b_1$, $l_2 \geq k_2$. The rest by induction.

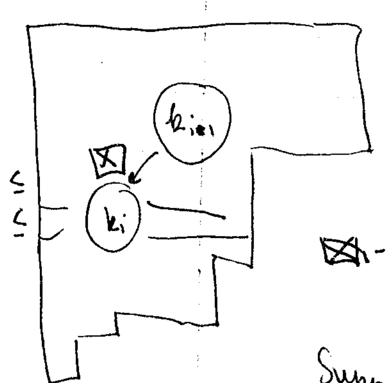
10/05/06

RSK Lemma



$\leftarrow k \leftarrow l$ if $k \leq l$ then $\boxed{Y}$ is located to the right of $\boxed{X}$

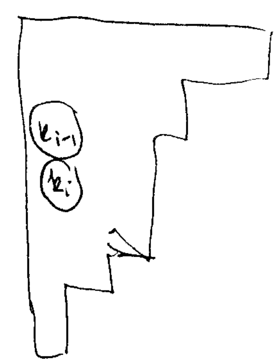
Lemma. If P is a SSYT then $P \leftarrow k$ is also a SSYT.



k_i is the first entry in ith row which is $> k_{i-1}$

$\boxed{X}$ - only problematic place, right above k_i

Suppose $x \geq k_{i-1}$ $\Rightarrow$ we have



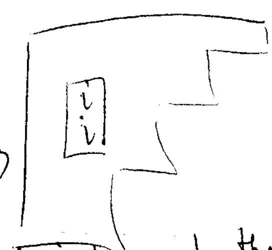
Clear by analysis of possibilities

Corollary. $w_A \xrightarrow{RSK} (P, Q)$ then both P, Q are SSYT.

For P is clear $\to$ just proved it.

Suppose we have $w_A \begin{pmatrix} \dots & i \; i \; i \dots i \; i & \dots \\ & \dots j_a \leq j_{a+1} \leq \dots \leq j_b \dots & \end{pmatrix}$

cannot.



$\Rightarrow Q$ is SSYT.

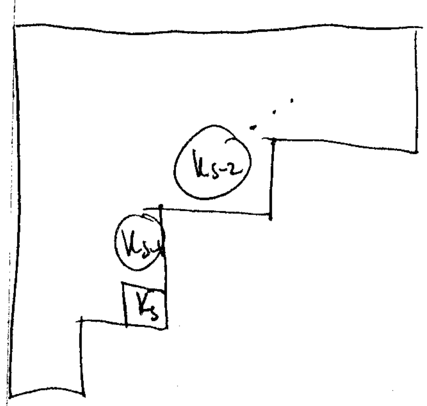
cannot bec insertion path is always to the right

Theorem. RSK is a bijection between matrices A and pairs (P, Q) of ~~Young tableau~~ SSYT of same shape. Moreover, ~~each j occurs~~ weight$(P) = (\beta_1, \beta_2, \dots)$ where β_i = ~~row~~ column sum of ~~row~~ colm. i, $\beta_j = \sum_i a_{ij}$ (j-th column sum)

weight$(Q) = (\gamma_1, \gamma_2, \dots)$ where $\gamma_i = \sum_j a_{ij}$ (i-th row sum)

Proof.

Moreover part is clear from construction. Just need to check this RSK is bijection only need to check that RSK is invertible.



$\forall$ SSYT, P with a marked corner $\boxed{k_s}$ we can reconstruct the insertion path.

k_{i-1} = the last entry in $(i-1)$st row s.t. $k_{i-1} < k_i$.

Inverse RSK

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \qquad Q = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

last inserted, since rightmost 3, which is max # in S

$$w_A = \begin{pmatrix} \dots & 2 & 3 & 3 \\ \dots & 2 & 1 & 2 \end{pmatrix}$$

Cauchy identity. $A \xrightarrow{RSK} (P, Q)$

$$\sum_A \left(\prod_{j \ge 1} x_j^{\sum_i a_{ij}}\right)\left(\prod_{i \ge 1} y_i^{\sum_j a_{ij}}\right) = \sum_\lambda \sum_{\substack{P, Q \\ \text{of shape } \lambda}} x^{wt(P)} y^{wt(Q)} = \sum_\lambda s_\lambda(x_1, x_2, \dots) s_\lambda(y_1, \dots)$$

$\parallel$ Cauchy $\rightarrow$ //

$$\sum_A \prod_{i,j} (y_i x_j)^{a_{ij}} = \prod_{i,j} \left(\sum_{a_{ij} \ge 0} (x_j y_i)^{a_{ij}}\right) = \prod_{i,j \ge 1} \frac{1}{1 - x_j y_i}$$

↑ all matrices

Properties of RSK

Theorem. If $A \xrightarrow{RSK} (P, Q)$ then $A^T \xrightarrow{RSK} (Q, P)$. In particular, for Schensted correspondence $w \mapsto (P, Q)$ (P, Q SYT) $\Rightarrow w^{-1} \mapsto (Q, P)$.

Lemma. Enough to prove "in particular" case.

Proof. RSK can be obtained from Schensted as follows:

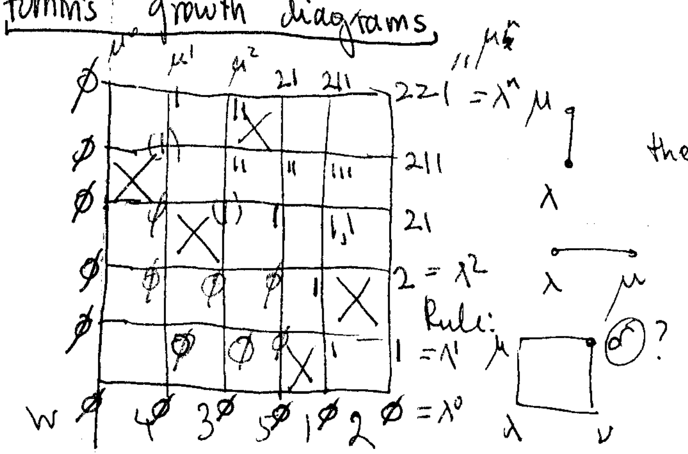
$$w_A \begin{pmatrix} 1 1 1 & 2 2 & 3 3 \\ 1 3 3 & 2 2 & 1 2 \end{pmatrix} \xrightarrow{RSK} \left(\begin{matrix} 1 1 2 2 \\ 2 3 \\ 3 \end{matrix} , \begin{matrix} 1 1 1 3 \\ 2 2 \\ 3 \end{matrix} \right)$$

from left to right

$$\begin{pmatrix} 1' 2' 3' & 4' 5' & 6' 7' \\ 1'' 6'' 7'' & 3'' 4'' & 2'' 5'' \end{pmatrix} \longrightarrow \left(\begin{matrix} 1'' 2'' 4'' 5'' \\ 3'' 7'' \\ 6'' \end{matrix} \quad \begin{matrix} 1' 2' 3' 7' \\ 4' 5' \\ 6' \end{matrix} \right)$$

claim: Schensted correspondence

Fomin's growth diagrams



then $\lambda = \mu$ or $\lambda \nearrow \mu$.

Rule:

μ, λ, ν, δ?

Local rules If there is no "X" in box then

(1) if $\lambda = \nu = \mu \Rightarrow \delta = \lambda$

(2) $\lambda \nearrow \mu = \nu \Rightarrow \mu$ is obtained by adding 1 to λ_i then δ is obtained from μ by adding 1 into μ_{i+1}.

(3) $\mu \neq \nu$ then $\delta = \mu \cup \nu$

If there is an "X" in the box then

(4) we'll see $\lambda = \nu = \mu$ and $\delta = (\lambda_1 + 1, \lambda_2, \ldots)$

$(\lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(n)}) \rightsquigarrow P$ $\qquad (\emptyset, 1, 2, 21, 211, 221)$

$(\mu^{(0)} \subset \mu^{(1)} \subset \quad \subset \mu^{(n)}) \rightsquigarrow Q$

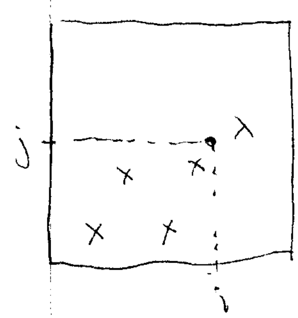
$$\downarrow \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \cline{1-1}\end{array} = P$$

$(\emptyset, 1, 11, 21, 211, 221) \to$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \cline{1-1}\end{array} = Q$$

<u>Theorem.</u> (1) Fillings of growth diagrams is well defined.

(2) $w \mapsto (P, Q)$ is the Schensted correspondence.



$w(i,j)$ = subsequence of ~~xxx~~ $w_1, \dots, w_i$ formed by elts $\leq$

Define the partition $\lambda(w(i,j)) = (\lambda_1, \lambda_2, \dots)$

λ_1 = length of a longest increasing subsequence in $w(i,j)$.

$\lambda_1 + \lambda_2$ = max # of elts that can be covered by 2 increasing subsequences of $w(i,j)$

$\lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_k = \dots\dots$ k increasing subseq...

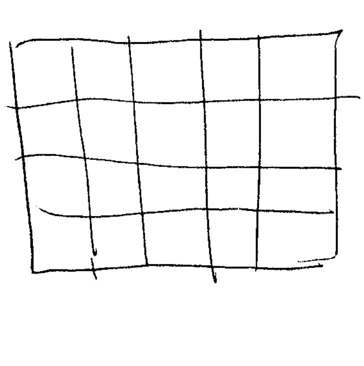
etc.

For diagram we had above $w(5,4) = (4,3,1,2)$

$\lambda_1 = 2$ $\qquad \lambda = (2,1,1)$

$\lambda_1 + \lambda_2 = 3$

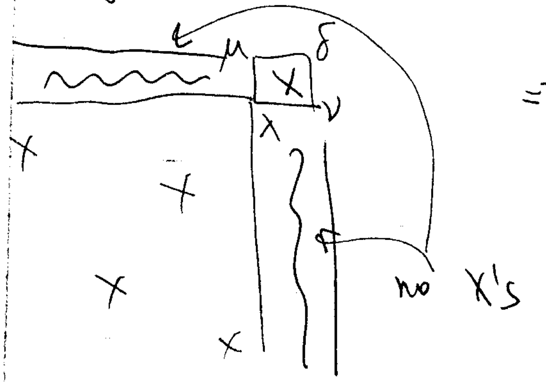
$\lambda_1 + \lambda_2 + \lambda_3 = 4$



Lemma. The collection of partitions $\lambda(w(i,j))$ satisfy the local rules.

$\rightarrow$ just need to check.

Case (4):



$\Rightarrow$ $\lambda = \nu = \mu$ since words will be exact

while for δ we have another X in the rectangle w/ δ in right upper corner

$\Rightarrow$ just increases length of longest increasing subseq. $\Rightarrow$

just add 1 to λ_1 to get δ.

Exercise

$$\begin{array}{ccccccccccc}
\lambda_1 & & \lambda_2 & & & \cdots & & & \lambda_n \\
& \geq x_1 \geq & & \geq x_2 \geq & \cdots & & & \geq x_{n-1} \geq & \\
& & \geq y_1 \geq & & & & \geq y_{n-2} & & \\
& & & & \geq z_1 \geq & & & &
\end{array}$$

Gelfand-Tsetlin patterns.

$GT(\lambda) \subseteq \mathbb{R}^{\binom{n}{2}}$ $\leftarrow$ # variables

polytope defined by these eqn. Calculate volume of this polyt

$$\begin{array}{ccccc}
5 & & 3 & & 2 \\
& \geq x_1 \geq & & \geq x_2 \geq & \\
& & \geq y_1 \geq & &
\end{array}$$