PROBLEM SET 4 (due on Tuesday 11/09/2004)

The problems worth 10 points each.

Problem 1 Let a_n be the sequence of integers given by the recurrence relation $a_n = 2a_{n-1} + 3a_{n-2}$, for $n \geq 2$, and $a_0 = a_1 = 2$. Find an explicit formula for the numbers a_n.

Problem 2 Find two constants b and c such that the sequence $a_n = 10^n - 2^n$ satisfies the recurrence relation $a_n = b\, a_{n-1} + c\, a_{n-2}$.

Problem 3 Let a_n be the sequence given by the recurrence relation $a_n = a_{n-2} + 1$, for $n \geq 2$, and $a_0 = a_1 = 1$. Find the ordinary and the exponential generating functions of this sequence. Can you give a combinatorial interpretation of the numbers a_n in terms of partitions of some kind?

Problem 4 Two sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are related by $b_n = \sum_{k=0}^{n} a_k$. What is the relationship between ordinary generating functions of these sequences?

Problem 5 Two sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are related by $b_n = \sum_{k\geq 0}^{\infty} \frac{a_{n+k}}{k!}$. (Assume that the sum converges.) What is the relationship between exponential generating functions of these sequences?

Problem 6 A *star* is a graph that contains a vertex adjacent to all edges of the graph. (In particular, the graph with a single vertex and the graph with a single edge are stars.) Let us say that a simple graph G is *stellar* if every connected component of G is a star. Let s_n be the number of stellar subgraphs of the complete graph K_n. We have $s_0 = 1$, $s_1 = 1$, $s_2 = 2$, $s_3 = 7$, $\ldots$. Find the exponential generating function $\sum_{n\geq 0} s_n \frac{x^n}{n!}$ for the numbers s_n.

Problem 7 Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ be the *Catalan number* and $C(x) = \sum_{n\geq 0} C_n x^n$. The *Motzkin number* M_n is defined as the number of paths from $(0,0)$ to $(n,0)$ that never go below the x-axis and are made of the steps $(1,0)$, $(1,1)$, and $(1,-1)$. We have $M_0 = 1$, $M_1 = 1$, $M_2 = 2$, $M_3 = 4, \ldots$ Let $M(x) = 1 + \sum_{n\geq 0} M_n x^{n+1} = 1 + x + x^2 + 2x^3 + 4x^4 + \cdots$. Show that $M(x) = C(x/(1+x))$.

Problem 8 Calculate the determinant of the almost upper-triangular $n \times n$-matrix

$$A = (a_{ij}), \text{ where } \qquad a_{ij} = \begin{cases} 2^{j-i+1} & \text{if } j - i + 1 \geq 0; \\ & \text{otherwise.} \end{cases}$$

Problem 9 Find a bijection between (unlabelled) plane binary trees with n leaves and (unlabelled) planar rooted trees with n vertices. (The numbers of these objects are the Catalan numbers.)

Problem 10 For a Catalan path P, let $h(P)$ be the number points in P located on the x-axis (excluding the initial and the final points $(0,0)$ and $(2n,0)$). Let $C_n(q)$ be the sum $\sum_P q^{h(P)}$ over Catalan paths of length $2n$. For example, $C_2(q) = 1+q$ and $C_3(q) = 2+2q+q^2$. In particular, $C_n(1)$ is the Catalan number. Find an explicit expression for the generating function $C(x,q) = \sum_{n\geq 0} C_n(q)\, x^n = 1 + x + (1+q)x^2 + (2+2q+q^2)x^3 + \cdots$ for these polynomials. Hint: Try to express $C(x,q)$ in terms of the generating function for the Catalan numbers.

Bonus Problems

Problem 11 (*) Calculate the determinant of the 5-diagonal $n \times n$-matrix

$$A = (a_{ij}), \text{ where } \qquad a_{ij} = \begin{cases} 1 & \text{if } i = j; \\ -1 & \text{if } i = j \pm 1; \\ 1 & \text{if } i = j \pm 2; \\ 0 & \text{otherwise.} \end{cases}$$

Problem 12 (*) Let M_n be the number of perfect matching m in the complete graph K_{2n} such that

- m contains no pair of crossing edges (i,k), (j,l), for $i<j<k<l$.
- m contains no pair of edges (i,j), $(i-1,j+1)$, for $i<j$.

Also let $\tilde{M}_n$ be the number of perfect matching $\tilde{m}$ in the complete graph K_{2n} such that

- $\tilde{m}$ contains no pair of nesting edges (i,l), (j,k), for $i<j<k<l$.
- $\tilde{m}$ contains no pair of edges (i,j), $(i+1,j+1)$, for $i<j$.

Show that $M_n = \tilde{M}_n$.

Problem 13 (*) (Kummer's theorem) Let m_n be the maximal power of 2 that divides the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Then $m_n + 1$ equals the sum of digits in the binary expansion of $n+1$.

For example, $C_6 = 132$ is divisible by 2^2 but not divisible by 2^3 because $6+1 = 111$ (binary) has 3 digits 1.