PROBLEM SET 2 (due on Thursday 10/07/2004)

The problems worth 10 points each.

Problem 1 Find the number of five-digit positive integers such that all digits are different. Assume that the first digit is not '0'.

Problem 2 Prove that $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$ using the Binomial theorem. Can you give a combinatorial proof, as well?

Problem 3 Find the minimal number of adjacent transpositions needed to obtain the permutation $2, 4, 7, 1, 5, 8, 3, 6$ from $8, 7, 1, 3, 5, 2, 4, 6$.

Problem 4 Show that $\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$, for $0 \leq k \leq n$.

Problem 5 Prove that the following 3 numbers are equals:
(A) the number of partitions of n into at most k parts,
(B) the number of partitions of $n + k$ into exactly k parts,
(C) the number of partitions of n into parts less than or equal to k.

Problem 6 Let $p(n)$ be the total number of partitions of n. Show that $p(n) - p(n-1)$ equals the number of partitions of n into parts greater than 1.

Problem 7 Find the number of partitions of 14 into 4 nonzero parts of different sizes.

Problem 8 Prove that the determinant

$$\begin{vmatrix} \binom{n}{k} & \binom{n}{k+1} & \binom{n}{k+2} \\ \binom{n}{k-1} & \binom{n}{k} & \binom{n}{k+1} \\ \binom{n}{k-2} & \binom{n}{k-1} & \binom{n}{k} \end{vmatrix}$$

is strictly positive, for any integer n and k such that $0 \leq k \leq n$.

Problem 9 Find the number of ways to subdivide the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ into 4 disjoint 2 element subsets? What about the set $\{1, \ldots, 2n\}$?

Problem 10 Let $F_n^{\geq k}$ be the number of compositions of n into parts greater than or equal to r, for some positive n and r. Show that these numbers satisfy the *generalized Fibonacci relation:*

$$F_n^{\geq r} = F_{n-1}^{\geq r} + F_{n-r}^{\geq r}, \quad \text{for } n > r.$$

Bonus Problems:

Problem 11 $(*)$ Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition whose Young diagram D_λ fits inside the $n \times n$-square, i.e, $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Recall that the number of rook placements in the diagram D_λ equals $R_\lambda = \lambda_n(\lambda_{n-1}-1)(\lambda_{n-2}-2)\cdots(\lambda_1 - n + 1)$, provided that all factors are strictly positive. (Otherwise, there is no way to place n nonattacking rooks in D_λ.)

Assume that $\lambda_{n-i} > i$, for $i = 0, \dots, n-1$. Let $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_n)$ be the *conjugate partition* to λ, i.e., its Young diagram $D_{\lambda'}$ is obtained from D_λ by reflection with respect to the main diagonal. Prove that

$$\{\lambda_n, \lambda_{n-1}-1, \lambda_{n-2}-2, \dots, \lambda_1-n+1\} = \{\lambda'_n, \lambda'_{n-1}-1, \lambda'_{n-2}-2, \dots, \lambda'_1-n+1\}.$$

as multisets. For example, for $\lambda = (6,5,5,3,2,2)$ we have $\lambda' = (6,6,4,3,3,1)$ and $\{2,1,1,2,1,1\} = \{1,2,1,1,2,1\}$. Can you present a permutation that transforms the first multiset to the second multiset?

What can you say if the condition $\lambda_{n-i} > i$ does not hold for some i?

Problem 12 $(*)$ Let us say that a permutation $w = w_1, w_2, \dots, w_n$ is *132-avoiding* if there is no triple of indices $i < j < k$ such that $w_i < w_k < w_j$. Recall that the *code* of the permutation w is the sequence $(c_1, \dots, c_n)$, where $c_i = \#\{j > i \mid w_j < w_i\}$, for $i = 1, \dots, n$.

(A) Show that the code of w is weakly decreasing $c_1 \geq c_2 \geq \cdots \geq c_n$ if and only if w is 132-avoiding.

(B) Find the number of 132-avoiding permutations in S_n.

Problem 13 $(*)$ A *royal rook* is a new type of chessman that can move as rook or king. In how many ways can 8 royal rooks be placed on an 8×8 chessbord so that no two royal rooks attack each other?

Problem 14 $(*)$ The *Bell number* $B(n)$ is the number of set partitions of the set $\{1, 2, \dots, n\}$ into a disjoint union of nonempty subsets. For example,

$$B(3) = \#\{123, 12|3, 13|2, 23|1, 1|2|3\} = 5.$$

(The bars '|' subdivide the set.) Show that the Bell number $B(n)$ equals the number of set partitions of the set $1, 2, \dots, n+1$ into a disjoint union of nonempty subsets such that no subset contains two adjacent numbers i and $i+1$. For example,

$$\#\{1|2|3|4, 13|2|4, 14|2|3, 24|1|3, 13|24\} = 5 = B(3).$$

Problem 15 $(*)$ For an $n \times k$ matrix, let us first arrange entries in each row in the weakly increasing order by permuting them; then arrange entries in each column in the weakly increasing order by permuting them. Show that entries of the resulting matrix weakly increase in both rows and columns.

Problem 16 $(*)$ Prove bijectively that, for any positive n and r, the number of compositions of $n + r - 1$ into parts greater than or equal to r is equal to the number of compositions of n into parts congruent to 1 modulo r.