18.218 Sprint 2017 — Problem Set 1

due Monday, April 3, 2017

Turn in as many problems as you want.

Problem 1. Let G be a simple graph (undirected, no loops, no multiple edges) on vertices $1, \ldots, n$. A *configuration* is a collection of nonnegative integers $c_1, \ldots, c_n$ assigned to the vertices of G.

We say that a vertex i of G is *unhappy* if

$$c_i < \frac{1}{2} \sum_{j \text{ is a neighbor of } i} c_j.$$

We also say that a vertex i is *excited* if

$$c_i > \frac{1}{2} \sum_{j \text{ is a neighbor of } i} c_j.$$

The *Sponsor Game* is the following game on configurations:

- Start with a configuration $(c_1, \ldots, c_n) = (0, \ldots, 0, 1, 0, \ldots, 0)$.
- Pick any vertex of i which is unhappy and add 1 to c_i.
- Stop if there are no unhappy vertices.

The *Excited Sponsor Game* is the following modification of the game:

- Start with the configuration $(c_1, \ldots, c_n) = (0, \ldots, 0)$.
- Pick any vertex of i which is not excited and add 1 to c_i.
- Stop if all vertices are exited.

(a) For both the Sponsor Game and the Excited Sponsor Game, show that, if there is a way to play the game so that it stops after N steps, then any way to play the game will produce the same result (the same final configuration) after N steps.

(b) Prove that, if G is a simply-laced Dynkin diagram (types ADE), then the Sponsor Game stops after finitely many steps.

(c) Prove that, if G is a simply-laced Dynkin diagram, then the Excited Sponsor Game stops after finitely many steps.

(d) Classify all graphs G for which the Sponsor Game stops.

(e) Classify all graphs G for which the Excited Sponsor Game stops.

Problem 2. *Kostant's Game* is the following game on configurations:

- Start with a configuration $(c_1, \ldots, c_n) = (0, \ldots, 0, 1, 0, \ldots, 0)$.

- Pick any vertex of i which is unhappy and replace c_i by
$$-c_i + \sum_{j \text{ neighbor of } i} c_j.$$
- Stop if there are no unhappy vertices.

Show that, if G is a simply-laced affine Dynkin diagram (i.e., an extended Dynkin diagram of type $\tilde{A}$, $\tilde{D}$, $\tilde{E}$), then there exists an infinite *periodic* way to play Kostant's Game, that is, the sequence of vertices where we apply the moves has the form $i_1, ..., i_N, i_1, \ldots, i_N, i_1, \ldots, i_N, \ldots$.

Problem 3. Let $A = (a_{ij})$ be a symmetric $n \times n$ matrix such that $a_{ij} \leq 0$, for any $i \neq j$.

Show that the following two conditions are equivalent:

(1) There exists an n-vector $v > 0$ such that $Av > 0$. (Here the notation $v > 0$ means that all entries of v are positive.)
(2) The matrix A is positive-definite, that is, all principal minors of A are positive.

Problem 4. Prove that, for every crystallographic root system, the root poset has a unique maximal element (the highest root). (If possible, try to avoid using the classification of root systems.)

Problem 5. Let $\mathcal{A}_0$ be the fundamental alcove of a root system of type A_r. ($\mathcal{A}_0$ is an r-dimensional simplex.) Find all isometries (i.e., distance preserving affine transformations $x \to Mx + b$ of the space V) that preserve the simplex $\mathcal{A}_0$.

Problem 6. For a crystallographic root system, prove that each alcove of the affine Coxeter arrangement contains exactly one point of the rescaled coroot lattice $\frac{1}{h}Q^\vee$ in its interior. (Here $h = ht(\theta) + 1$ is the Coxeter number.)

Problem 7. (a) Let I be a subset of $\{(i, j) \mid 1 \leq i < j \leq n\}$. Prove that I is the set of inversions $Inv(w) := \{(i, j) \mid 1 \leq i < j \leq n, w(i) > w(j)\}$ of a permutation $w \in S_n$ if and only if, for any $i < j < k$, the set I satisfies:

(1) if (i, j) and (j, k) are in I, then (i, k) is in I.
(2) If (i, j) and (j, k) are not in I, then (i, k) is not in I.

(b) For any crystallographic root system Φ, prove that a subset I of positive roots Φ^+ is the inversion set $Inv(w) := \{\alpha \in \Phi^+ \mid w(\alpha) \notin \Phi^+\}$ of an element of the Weyl group $w \in W$ if and only if, for any triple of positive roots $\alpha, \beta, \gamma \in \Phi^+$ such that $\alpha + \gamma = \beta$, the set I satisfies:

(1) If α and γ are in I, then β is in I.

(2) If α and γ are not in I, then β is in not I.

Problem 8. Let us label the boxes of the staircase Young diagram $\lambda = (n-1, n-2, \ldots, 1)$ by pairs (i, j), $1 \le i < j \le n$, as follows:

$$\begin{array}{llllll}
(1,n) & (2,n) & (3,n) & \cdots & (n-2,n) & (n-1,n) \\
(1,n-1) & (2,n-1) & (3,n-1) & \cdots & (n-2,n-1) & \\
\vdots & \vdots & \vdots & \ddots & & \\
(1,4) & (2,4) & (3,4) & & & \\
(1,3) & (2,3) & & & & \\
(1,2) & & & & &
\end{array}$$

A *balanced tableau* T of the staircase shape $\lambda = (n-1, n-2, \ldots, 1)$ if a filling of the Young diagram λ by the numbers $1, 2, \ldots, N = \binom{n}{2}$ (without repetitions) such that, for any $i < j < k$ in $[N]$, the entries a, b, c of the boxes (i, j), (i, k), (j, k) in T satisfy $a < b < c$ or $a > b > c$.

Prove that the following construction gives a bijection between reduced decompositions $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$ of the longest permutation w_0 in the symmetric group S_n and balanced tableaux T of the shape $\lambda = (n-1, \ldots, 1)$.

Let $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N} = s_{k_N l_N} \cdots s_{k_2 l_2} s_{k_1 l_1}$, where, for $a = 1, \ldots, N$, $s_{k_a l_a}$ is the transposition of $k_a < l_a$ given by

$$s_{k_a l_a} = s_{i_1} s_{i_2} \cdots s_{i_{a-1}} s_{i_a} s_{i_{a-1}} \cdots s_{i_2} s_{i_1}.$$

Then, for $a = 1, \ldots, N$, the entry of the box (k_a, l_a) in T is a.

Problem 9. Generalize the previous problem to any root system Φ (and prove it).

Problem 10. For any group G and $m \ge 2$, the *Hurwitz action* is the action on m-tuples $(g_1, \ldots, g_m)$ of elements of G generated by the generators σ_i, $i = 1, \ldots, m-1$, given by

$$\sigma_i : (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \ldots, g_m).$$

Assume that $G = S_n$ (the symmetric group) and $m = n - 1$. Let $s_1, s_2, \ldots, s_{n-1}$ be the simple transpositions in S_n (with the standard indexing). Prove that the number of $(n-1)$-tuples in S_n obtained from from $(s_1, \ldots, s_{n-1})$ by the Hurwitz action equals the number n^{n-2} of spanning trees of the complete graph K_n.

For example, for $n = 3$, we obtain $3 = 3^{3-2}$ pairs (s_1, s_2), $(s_1 s_2 s_1, s_1)$, $(s_2, s_2 s_1 s_2)$.

Problem 11. A *non-crossing tree* in K_n (with vertices labelled $1, 2, \ldots, n$) is a tree without a pair of edges (i, j) and (k, l) such that $i < k < j < l$.

For any *ordered* $(n-1)$-tuple $(g_1, \dots, g_{n-1})$ of transpositions in S_n obtained from $(s_1, \dots, s_{n-1})$ by the Hurwitz action (see the previous problem), consider the unordered $(n-1)$-tuple $\{g_1, \dots, g_{n-1}\}$ and identify it with edges of a subgraph in K_n. (The transposition of i and j correspondes to an edge (i, j).)

(a) Prove that all subgraphs of K_n obtained by this procedure are exactly all non-crossing trees.

(b) Find a formula for the number of non-crossing trees in K_n.

Problem 12. (a) For two permutations $u, w \in S_n$, show that $u \le w$ in the weak Bruhat order on S_n if and only if $Inv(u^{-1}) \subseteq Inv(w^{-1})$.

(b) For two permutations $u, w \in S_n$, show that $u \le w$ in the strong Bruhat order on S_n if and only if $r_{ij}(u) \ge r_{ij}$, for any $i, j \in [n]$, where

$$r_{ij}(w) := \#\{k \mid 1 \le k \le i, w(i) \le j\}.$$

Problem 13. A permutation $w \in S_n$ is called *fully commutative* if all reduced decompositions of w are obtained from each other by using a sequence of the commutation relations $s_i s_j = s_j s_i$, for $|i - j| \ge 2$.

Show that the symmetric group S_n contains exactly the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ of fully commutative elements.

Problem 14. An *upper order ideal* in the root poset $(\Phi^+, \le)$ is a subset $I \subset \Phi^+$ such that if $\alpha \in I$ and $\beta \ge \alpha$ then $\beta \in I$.

An upper order ideal I in the root poset is called an *abelian ideal* if I does not contain a triple of roots α, β, γ such that $\beta = \alpha + \gamma$.

(a) For type A_{n-1}, show that the number of upper order ideals in the root poset equals the Catalan number C_n.

(b) For type A_{n-1}, show that the number of abelian ideals in the root poset equals 2^{n-1}.

(c) For any crystallographic root system Φ of rank r, show that the number of abelian ideals in Φ^+ equals 2^r.