18.217 Problem Set 2

due Friday, December 4, 2020

① (a) Prove that $(x_1+x_2+\ldots)^n$ equals $\sum_{\lambda \vdash n} f_\lambda \, S_\lambda(x_1, x_2, \ldots)$, where f_λ is the number of SYTs of shape λ.

(b) Prove that $(x_1+x_2+\ldots)^n$ equals the sum of skew Schur functions $S_{\lambda/\mu}$ over all 2^{n-1} ribbons with n boxes.

For example, $(x_1+x_2+\ldots)^3 =$

$$= S_{(3)} + 2\, S_{(2,1)} + S_{(1,1,1)}$$

$$= S_{(3)} + S_{(2,1)} + S_{(2,2)/(1)} + S_{(1,1,1)}.$$

(2) Express the sum of all 4-cycles $\sum (a,b,c,d) \in \mathbb{C}[S_n]$ in terms of the Jucys–Murphy elements $X_1, X_2, \ldots, X_n \in \mathbb{C}[S_n]$

$$X_i := \sum_{j \in \{1, \ldots, i-1\}} (j, i).$$

(3) For a marked partition

$$\overline{c_1} + c_2 + \ldots + c_l = n,$$

let $[\overline{c_1}, \ldots, c_l] \in \mathbb{C}[S_n]$ be the sum of all permutations $w \in S_n$ with cyclic type $(c_1, \ldots, c_l)$ such that the index n belongs to the cycle of size c_1.

Show that each element $[\overline{c_1}, \ldots, c_l]$ can be expressed in terms of elements of the form $[\overline{1}, \tilde{c}_2, \ldots, \tilde{c}_l]$ and the Jucys–Murphy element X_n.

④ Show that for any two standard Young tableaux of the same shape, one can obtain one tableau from the other by a sequence of the following operations:

Exchange the entries i and $i+1$ in the tableau if these two entries <u>are not</u> in the same row or the same column.

Example,

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \longleftrightarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \longleftrightarrow \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$$

$$\updownarrow$$

$$\text{etc.} \longleftrightarrow \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

⑤ Calculate all values of the character $\chi_{(n-1,1)}$ of the irreducible representation of S_n (Specht module) associated with the Young diagram

$$\lambda = (n-1, 1) = \begin{array}{|c|c|c|c|}\hline \ & \ & 1 \cdots & \ \\ \hline \ \\ \cline{1-1}\end{array}.$$

(6) For a sequence of integers $i_1, \dots, i_n$, let

$$\langle i_1, \dots, i_n \rangle := \frac{1}{x_{i_1} - x_{i_2}} \cdot \frac{1}{x_{i_2} - x_{i_3}} \cdots \frac{1}{x_{i_{n-1}} - x_{i_n}}.$$

(a) Let A, B be two nonempty disjoint sequences of indices in $\{2, \dots, n\}$ without repeated entries.

Prove that

$$\sum_{C \in \text{Shuffle}(A,B)} \langle 1, C, 1 \rangle = 0,$$

where Shuffle(A,B) is the set of all shuffles of sequences

Example, for $A = \{2\}$, $B = \{3,4\}$ we have

$$\langle 1,2,3,4,1 \rangle + \langle 1,3,2,4,1 \rangle + \langle 1,3,4,2,1 \rangle = 0.$$

(b) (Kleiss-Kuif relations) Let A, B be two disjoint sequences of indices in $\{2, \dots, n-1\}$ without repeated entries.

Prove that

$$\langle 1, A, n, B, 1 \rangle = (-1)^{|B|} \sum_{C \in \text{Shuffle}(A, B^T)} \langle 1, C, n, 1 \rangle,$$

where B^T is the reversed sequence B.

⑦ In class, we gave 2 bijective proofs of the identity:

$$\sum_{\substack{\tilde{\mu} : \mu \subset \tilde{\mu} \subseteq \lambda \\ \tilde{\mu}/\mu \text{ is a horizontal} \\ k\text{-strip}}} s_{\lambda/\tilde{\mu}} = \sum_{\substack{\tilde{\lambda} : \mu \subseteq \tilde{\lambda} \subset \lambda \\ \lambda/\tilde{\lambda} \text{ is a horizontal} \\ k\text{-strip}}} s_{\tilde{\lambda}/\mu}$$

for any skew Young diagram λ/μ and a positive integer k such that $k \leq |\lambda/\mu|$.

One bijection is based on jeu-de-taquin and the other bijection is based on the Bender-Knuth involution.

Are these two bijections equal to each other?

(8) Let $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ be the Bender-Knuth involution (acting on SSYT's).

Define the <u>twisted Bender-Knuth involutions</u> by

$$\tilde{\sigma}_i := q_i^{-1} \sigma_1 q_i ,$$

where

$$q_i := (\sigma_1 \sigma_2 \ldots \sigma_i)(\sigma_1 \sigma_2 \ldots \sigma_{i-1}) \ldots$$
$$\ldots (\sigma_1 \sigma_2)(\sigma_1).$$

Prove that the twisted Bender-Knuth involutions $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{n-1}$ satisfy the Coxeter relations:

- $(\tilde{\sigma}_i) = 1$
- $\tilde{\sigma}_i \tilde{\sigma}_j = \tilde{\sigma}_j \tilde{\sigma}_i , \quad |i-j| \geq 2$
- $\tilde{\sigma}_i \tilde{\sigma}_{i+1} \tilde{\sigma}_i = \tilde{\sigma}_{i+1} \tilde{\sigma}_i \tilde{\sigma}_{i+1}$

Also check that the original Bender-Knuth involutions $\sigma_1, \ldots, \sigma_{n-1}$ <u>do not</u> satisfy the Coxeter relations.

(9) Prove the identity

$$\sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda = h_n$$

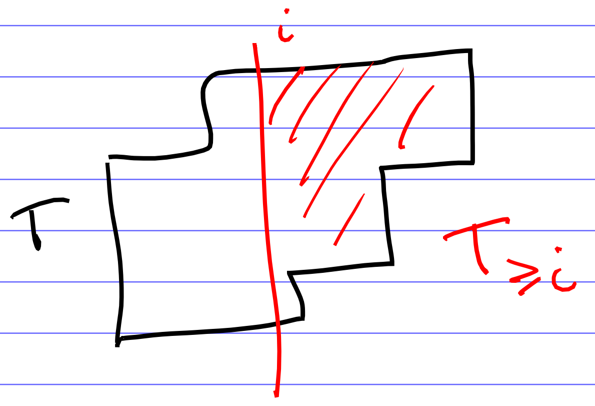
(10) Construct a bijection between the following two sets of Littlewood-Richardson tableaux $LR(\lambda/\mu, \nu)$ and $LR((\lambda/\nu)^\vee, \mu)$,

where $(\lambda/\nu)^\vee$ denotes the skew Young diagram λ/ν rotated by $180°$.

(11) Construct a bijection between the set of Littlewood-Richardson tableaux $LR(\lambda/\mu, \nu)$ and the set of Zelevinsky's pictures for λ/μ and ν.

The following two problems are related to the lecture on "Stembridge's concise proof of the LR-rule".

(12) Prove that a semi-standard Young tableau T of shape λ/μ and weight ν is a Littlewood-Richardson tableau (i.e., T satisfies the lattice word condition) if and only if, for any $j \geq 1$, the weight of the tableau $T_{\geq j}$ is a partition, where $T_{\geq j}$ denotes the subtableau of T in columns $\geq j$.



(13) Show that specialization for $\nu = \varnothing$ of Zelevinsky's formula

$$S_\lambda \cdot S_{\mu/\nu} = \sum_\gamma c(\lambda, \mu/\nu, \gamma) S_\gamma,$$

where $c(\lambda, \mu/\nu, \gamma)$ is the number of SSYT's T of shape μ/ν and weight $\gamma - \lambda$ such that, $\forall j \geq 1$ $\lambda + \text{weight}(T_{\geq j})$ is a partition

is equivalent to the classical Littlewood-Richardson rule.

(14) For a SSYT tableau T (of a skew shape), let a_{ij} denotes the number of enries "j" in the i^{th} row of T.

Explicitly, express the "lattice word condition" by linear inequalities for a_{ij}'s.

(In class, we wrote the first few inequalities:

$a_{11} \geq a_{22}$, $a_{22} \geq a_{33}$,

$a_{11} + a_{21} \geq a_{22} + a_{32}$, etc.)

(15) In class, we showed how to transform a BZ-triangle into a Knutson-Tao's puzzle.

Rigorously prove that this gives a bijection between BZ-triangles and puzzles.

(16) Prove using honeycombs that, if $c_{\lambda\mu}^{\nu} \neq 0$, then

$$\lambda_{i+1} + \mu_{j+1} \geq \nu_{i+j+1} \quad \text{for}$$

$i, j \geq 0$.

(17) Find 3 partitions λ, μ, ν such that the Berenstein-Zelevinsky polytope $BZ(\lambda, \mu, \nu)$ has a non-integer vertex.

Draw the honeycomb that corresponds to this vertex of $BZ(\lambda, \mu, \nu)$.

(18) Let $c_{\mu\nu}^{\lambda}$ be the Littlewood-Richardson coefficients defined by

$$s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda}\, s_{\nu}.$$

Prove that the Schur function $s_\lambda(x_1, x_2, \ldots, y_1, y_2, \ldots)$ in two infinite sets of variables $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ expands, as follows:

$$s_\lambda(x_1, x_2, \ldots, y_1, y_2, \ldots) =$$

$$= \sum_{\mu, \nu} c_{\mu\nu}^{\lambda}\, s_\mu(x_1, x_2, \ldots)\, s_\nu(y_1, y_2, \ldots).$$

(19) Prove the hook length formula for <u>shifted</u> shapes (formulated in class).