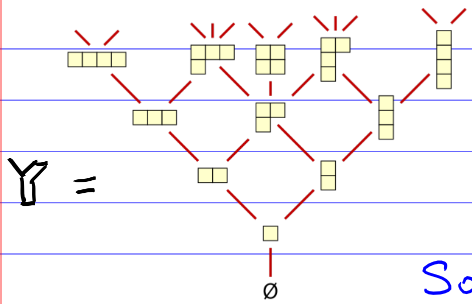
18.217 Lecture 8 09/21/2020

Young lattice $\mathbb{Y}$ = the poset of Young diagrams ordered by inclusion.

$\mathbb{Y}$ = 

$\mathbb{Z}[\mathbb{Y}]$ = the space of formal linear combinations of Young diagrams.

Some elements of $\mathbb{Z}[\mathbb{Y}]$:

$(2)\ ,\quad 27\cdot\varnothing + 3\,(2,1) - (1,1)$

Up & Down operators acting on $\mathbb{Z}[\mathbb{Y}]$

$$U:\ \lambda \longmapsto \sum_{\mu:\ \mu \gtrdot \lambda} \mu$$

$$D:\ \lambda \longmapsto \sum_{\mu:\ \mu \lessdot \lambda} \mu$$

" $\gtrdot$ " covering relation in $\mathbb{Y}$

Ex

$U:\ (2) \longmapsto (3) + (2,1)$

$D:\ (2,1) \longmapsto (2) + (1,1)$

$(D\cdot U):\ (2) \longmapsto 2\,(2) + (1,1)$

$(U\cdot D):\ (2) \longmapsto (2) + (1,1)$

$(DU-UD):\ (2) \longmapsto (2)$

Lemma $\boxed{(DU-UD) = I}$

identity operator

commutator $[D,U] := DU - UD$

Proof

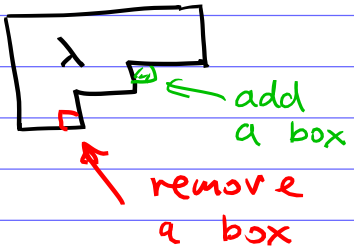
$$\begin{tikzcd} & \mu \arrow[dr, "\text{down}"] & \\ \lambda \arrow[ur, "\text{up}"] \arrow[dr, "\text{down}"'] & & \nu \\ & \tilde{\mu} \arrow[ur, "\text{up}"'] & \end{tikzcd}$$

The coeff. of ν in $[D, U](\lambda)$

$=$ # such μ's $-$ # such $\tilde{\mu}$'s.

2 cases:

(I) $\lambda \neq \nu$



$\lambda \xrightarrow{U} \mu \xrightarrow{D} \nu$

We first add a box to λ and then remove a different box.

We can do these operations in a different order (first remove & then add) to get the same result.

So the coeff. of ν in $[D, U](\lambda) = 0$

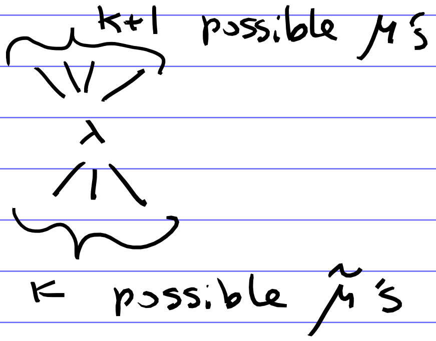
(if $\nu \neq \lambda$)

(II) $\lambda = \nu$.

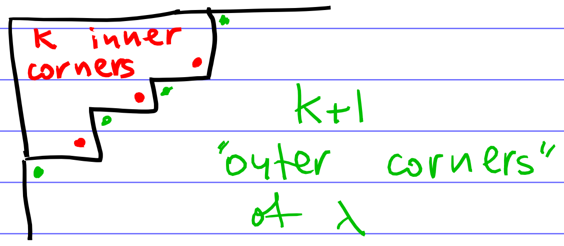
{ways to add a box to λ & then remove the same box}

$-$ # {ways to remove a box from λ & then add the same box}

In $\mathbb{Y}$:



Why?

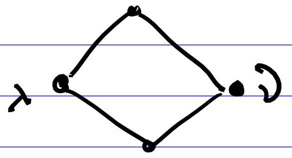


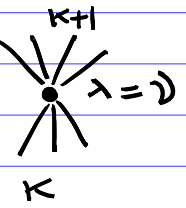
So the coeff. of λ in

$$[D,U](\lambda) = (k+1) - k = 1$$

We obtain $[D,U](\lambda) = \lambda$.

The properties of $\mathbb{Y}$ "responsible" for th identity $[D,U] = I$:

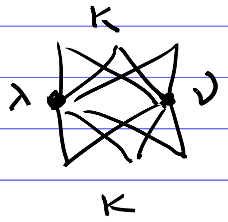
I.  $\lambda \neq \nu$

II. 

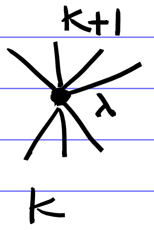
Def. A <u>differential</u> <u>poset</u> is a ranked poset with a unique minimal elt. $\hat{0}$ s.t.

$[D, U] = I$ holds for the up & down oper. on $\mathbb{Z}[P]$.

$$\Updownarrow$$

(I) 

$\forall\ \lambda \neq \nu$ on the same level

(II)  $\forall\ \lambda$

So $\mathbb{Y}$ is a "prototipical" differential poset.

Why "differential"?

Because the operators

$X : f(x) \longmapsto x f(x)$ &

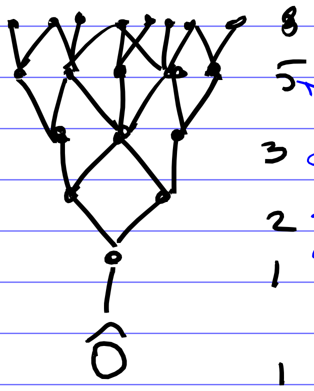
$\frac{d}{dx} : f(x) \longmapsto f'(x)$ acting on $\mathbb{Z}[x]$

satisfy the same

relation $\left[\frac{d}{dx}, X\right] = I.$

Another differential poset

<u>Fibonacci lattice</u>:

$\mathbb{F} =$ 

rank #'s of $\mathbb{F}$ are the Fibonacci numbers.

- Start with 0-th level $\{\hat{0}\}$
- Then recursively construct level k by "reflecting" the covering relations between levels $k-1$ & $k-2$ and then adding elements covering elts. on level $k-1$.

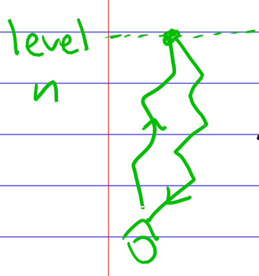
<u>Remark</u> $\mathbb{F} \neq \mathbb{Y}$ because the Fibonacci numbers $F_{n+1} \neq$ the partition numbers $p(n)$, for $n \geq 5$:

n	0	1	2	3	4	5	6	7	8 ...
F_{n+1}	1	1	2	3	5	8	13	21	34
$p(n)$	1	1	2	3	5	7	11	15	22 ...

Theorem For any differential poset (e.g. for $\mathbb{Y}$ or $\mathbb{F}$) we have

$$D^n U^n \hat{0} = n!\,\hat{0}$$

this is the min. elt. of P, not 0



Equiv.
$$\sum_{\substack{\lambda \text{ elt. of } P\\ \text{of rank } n}} \#\{\text{saturated chains } \hat{0}\to\dots\to\lambda\}^2 = n!$$

$$\Big(\text{For } \mathbb{Y}: \sum_{\substack{\lambda \text{ Young}\\ \text{diagram st.}\\ |\lambda|=n}} f_\lambda^{\,2} = n!\Big)$$

Proof. This follows from

(1) $[D, U] = I$

(2) $D\,\hat{0} = 0$

there is no way to go down from the min. elt.

$$\boxed{DU = UD + I}$$

Ex. $n=2$

$$DDUU(\hat{0}) = DUDU(\hat{0}) + DU(\hat{0})$$

D either "jump" over U to the right, or "annihilates" with U

$$= (\underbrace{DUUD}_{=0} + DU + \underbrace{UD}_{=0} + I)(\hat{0})$$

$$= (\underbrace{UD}_{=0} + I + I)(\hat{0})$$

$$= 2\,\hat{0}.$$

Each D moves to the left by "jumping over" U's untill it "annihilates" with some U. (It cannot jump over all U's, because we would get

$$\dots D\; U \dots U\, (\hat{0}) \rightsquigarrow \dots U \dots U\, \underbrace{D(\hat{0})}_{=0}$$

There are $n!$ ways to match n D's with n U's into n pairs of D & U which annihilate each other

Ex. $D\ D\ D\ D\ U\ U\ U\ U\ (\hat{0})$

Another argument: The operators $X : f \mapsto x f$ & $\frac{d}{dx}$

satisfy the same relations

- $\left[\frac{d}{dx}, X\right] = 1$
- $\frac{d}{dx}(1) = 0$

So the calculation $D^n U^n (\hat{0}) = ?\ \hat{0}$ is equivalent to

$$\left(\frac{d}{dx}\right)^n X^n (1) = \left(\frac{d}{dx}\right)^n (x^n) = n! \cdot 1$$

$$\Rightarrow\ ? = n! \qquad \square$$

How about more general paths in $\mathbb{Y}$ (or in any diff. poset)?

Consider any word w with n U's & n D's.

<u>Ex.</u> $w = DDDUUDUU$

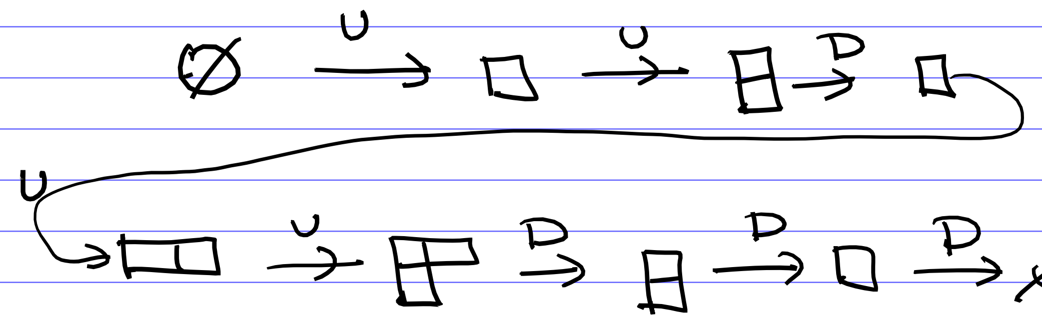
$DDDUUDUU(\emptyset) = ?\,\emptyset$

$? = \#\{\text{oscillating tableaux}\}$

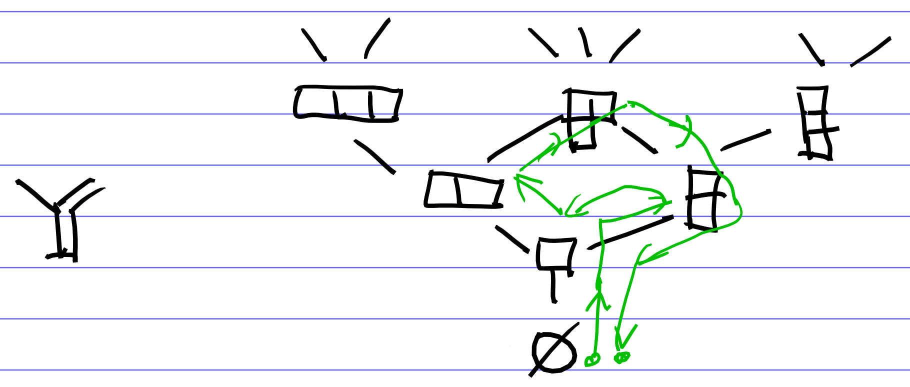
<u>Def.</u> An <u>oscillating tableaux</u> is any path going along the edges of Hasse diagram of $\mathbb{Y}$ with fixed initial & final Young diagram & prescribed sequence of Up & Down steps given by word w in U's & D's (read backword).

In our case, the initial & final Young diagrams are $\emptyset$, and w consists of n U's and n D's.

<u>Example.</u> An oscillating tableau



$w = DDDUUDUU$.



The same argument as before shows that

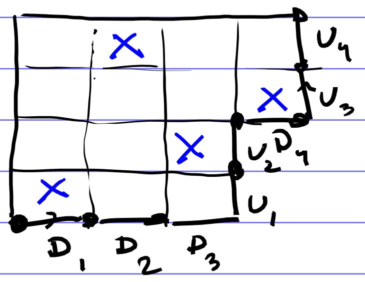
$? = \#$ ways to match all D's with all U's s.t. each D is matched with a U to the <u>right of D</u>

<u>Ex.</u> $w = D_1 D_2 D_3 U_1 U_2 D_4 U_3 U_4$

$? = \#$ such matchings.

Such matchings correspond to **rook placements**

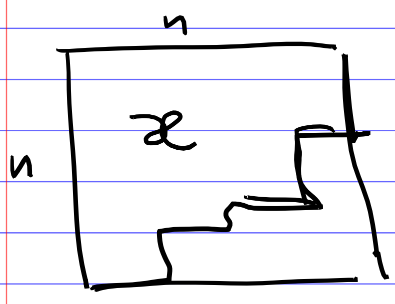
Ex.



If D_i is matched with U_j, then we place a rook in column i & row j (labelled from the bottom)

The rooks should not attack each other.

Theorem Let $\varkappa = (\varkappa_1, \varkappa_2, \dots, \varkappa_n)$, $n \geq \varkappa_1 \geq \dots \geq \varkappa_n \geq 0$ Young diagram that fits inside $n \times n$ square.



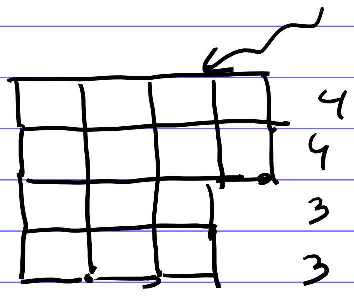
Then # ways to place n non-attacking rooks into the shape $\varkappa$ equals

$$\#\begin{Bmatrix}\text{rook} \\ \text{placements}\end{Bmatrix} =$$

$$= \varkappa_n(\varkappa_{n-1}-1)(\varkappa_{n-2}-2)\dots(\varkappa_1-n+1)$$

or 0 if some term is ≤ 0.

Example. $w = DDDUUDUU$



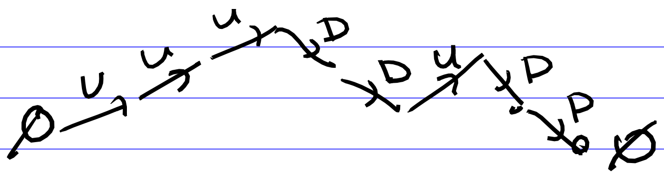
$\varkappa = (4,4,3,3)$

rook placements

$= 3 \cdot 2 \cdot 2 \cdot 1$

$= 12$

$\Rightarrow$ # oscillating tableaux s.t.



$= 12$

If $W = \underbrace{D \ldots D}_{n} \underbrace{U \ldots U}_{n}$

$\mathcal{X}$ = $n \times n$ square.

$\#\{$placements of n rook in $n \times n$ square$\} = n!$

Now we know that

$\#\{$oscillating tableaux$\}$

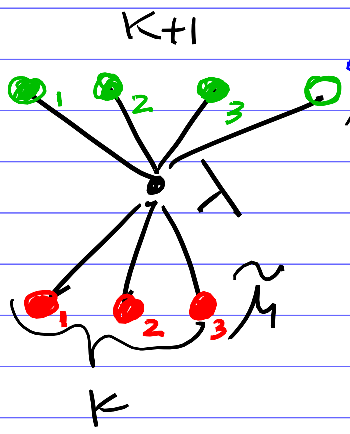
$= \#\{$rook placements$\}$

i.e. paths in $\mathbb{Y}$ from $\emptyset$ to $\emptyset$ with particular sequence of up & down steps

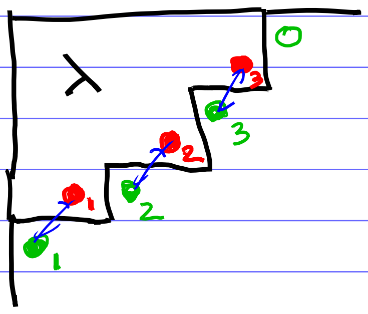
Q. Can we construct a bijection between oscillating tableaux and rook placements?

For any $\lambda \in \mathbb{Y}$, we need to fix the following correspondence between all k μ's covering λ (except 1) and k $\tilde{\mu}$ covered by λ.

$\mathbb{Y}$:



one special "unmatched" μ



- labell the inner/outer corners of λ by $1, 2, \ldots, k$
- Match i^{th} inner corner with i^{th} inner corner
- leave the rightmost outer corner unmatched.

Bijection $\varphi_w : \{\text{oscillating tableaux}\}$
$\downarrow$
$\{\text{rook placements}\}$.

w is a word in U's & D's

Given:
T - an oscillating tableau from $\emptyset$ to $\emptyset$ of "shape" given by word w,

$\mathcal{X} \subseteq n\times n$ square the Young diagram corr. to w.

Want to construct rook placement $R = \varphi_w(T)$ in $\mathcal{X}$.

We'll construct φ_w by induction.

Base if w = empty word

Then $\varphi_\emptyset$: empty osc. tabl. $\longleftrightarrow$ empty rook placement

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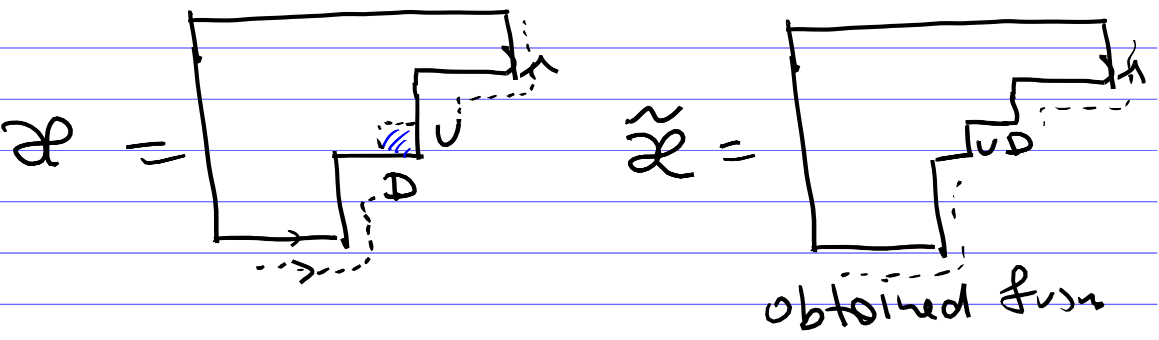
Step of Induction:

Find any fragment "DU" in w

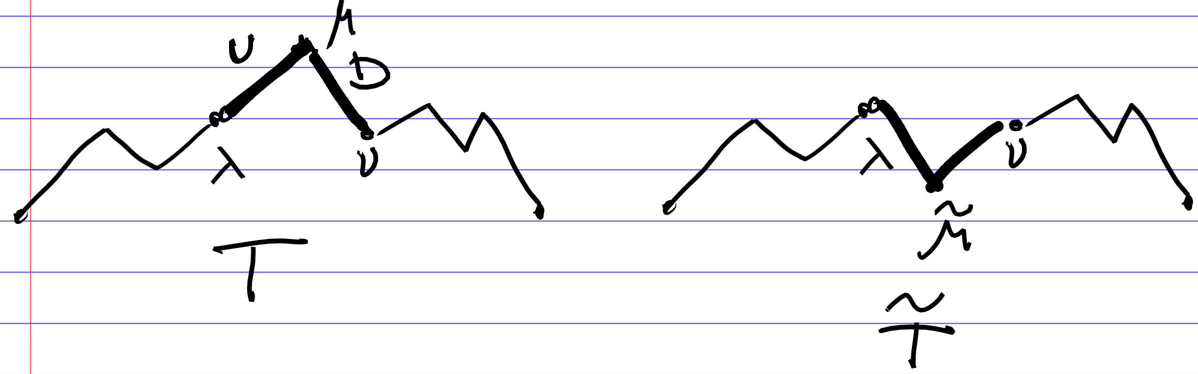
$w = \ldots\ \text{D U} \ldots$, let
$\tilde{w} = -\|-\ \text{U D} -\|-$ (switch these U & D)

For corresponding shapes, we have



obtained from $\mathcal{X}$ by removing the corner box

In the oscillating tableau T, we have



In all cases, we have a correspondence between possible μ's and $\tilde{\mu}$'s, except the case when
- $\lambda = \nu$
- μ is obtained from λ by adding a box in 1^{st} row, (special outer corner box)

Then $\varphi_w(T) = \varphi_{\tilde{w}}(\tilde{T})$

in non-exceptional cases

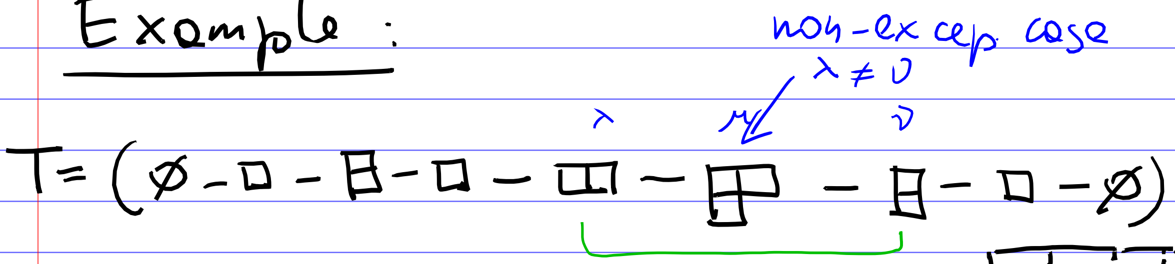
In the exceptional case

- Place the rook in box of x
- $\tilde{w}$ obtained from w by removing this fragment "DU"

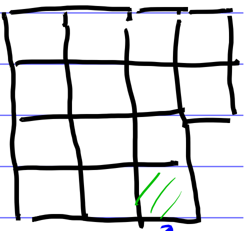
and $\tilde{T}$ obtained from T by removing this up & down steps.

- Place all other rooks as in $\varphi_{\tilde{w}}(\tilde{T})$

Example:

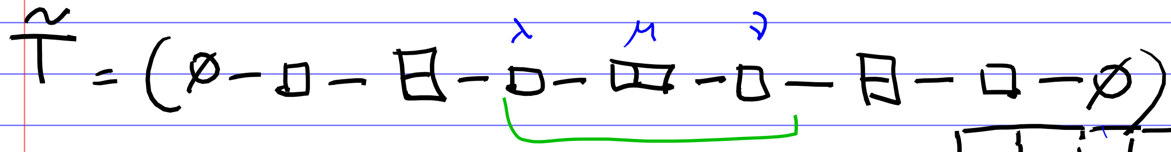


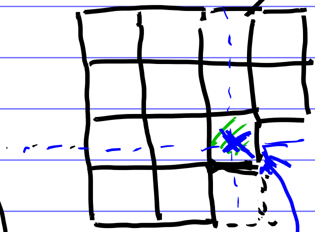
$w = DDD\underline{U}UUDUU \rightsquigarrow x =$



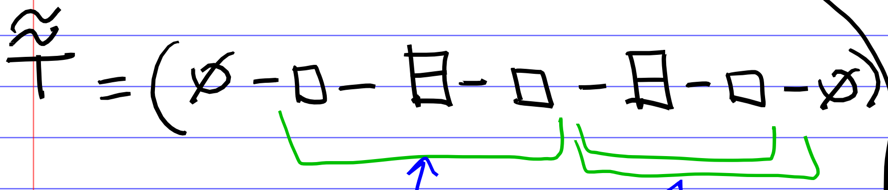
no rook here

exceptional case:
- $\lambda = \nu$ and
- μ/λ has box in 1^{st} row

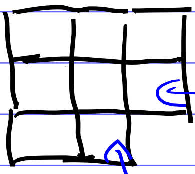




place a rook in this box

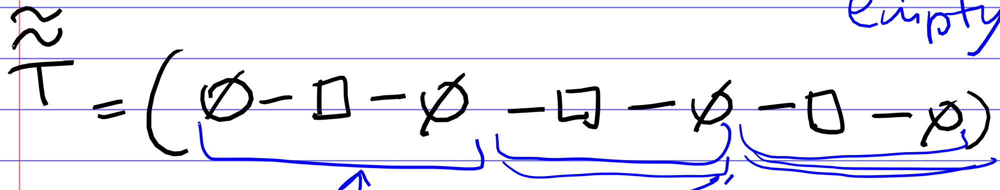


non-exceptional cases

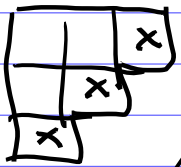


empty

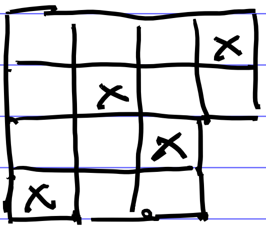
empty



exceptional



We got $\varphi_w(T) =$



Theorem. We've got a bijection $\left\{\begin{matrix}\text{osc.}\\ \text{tableaux}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{rook}\\ \text{placements}\end{matrix}\right\}$

It is clear from the construction that it is symmetric:

T osc. tableau

$\mathrm{rev}(T)$ is reversal (reverse the path)

if $T \mapsto R$

$\mathrm{rev}(T) \mapsto \mathrm{transpose}(R)$

Special Case $w = \underbrace{D \dots D}_{n}\underbrace{U \dots U}_{n}$

$\mathcal{X} = n \times n$ square

In this case, this constr. gives Schensted corresp.

$(P, Q) \xleftrightarrow[\text{Schensted}]{\sim}$ permutation w

pair of SYT's viewed as an oscillating tableau

Corollary. Schensted corresp. is symmetric:

if $(P, Q) \longleftrightarrow w$

then $(Q, P) \longleftrightarrow w^{-1}$
