18.217 Lecture 7 09/18/2020

RSK (cont'd)

Insertion Algorithm

P - a semi-standard Young tableau

j - positive integer

$P \leftarrow j$: the SSYT w/ 1 more box, obtained as follows:

Algorithm

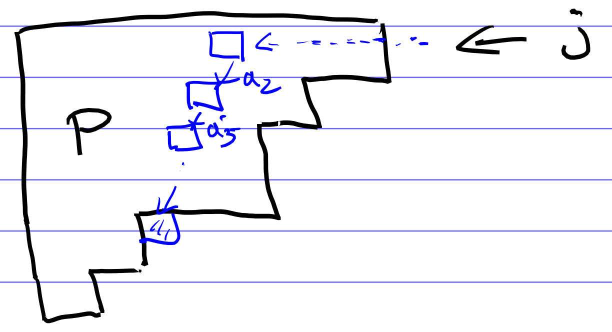
(1) $r := 1$ (row index), $a_r := j$

(2) IF $a_r \geq$ largest entry of row r, then add 1 box in the end of row r filled with a_r, and STOP.

(3) OTHERWISE, find the smallest entry a_{r+1} of row r s.t. $a_{r+1} > a_r$. Replace a_{r+1} with a_r

(4) Set $r := r+1$

(5) GO TO (2)



Ex.

$j = 2$

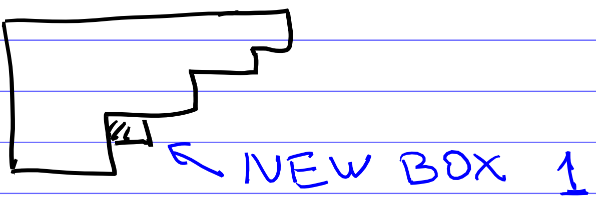
$P =$

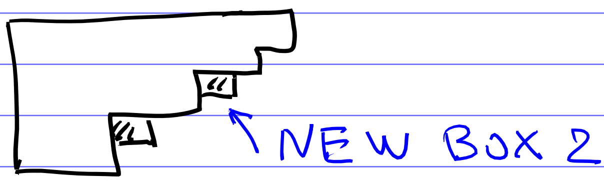
1	1	1	2	3	3	5
2	2	2	3	4	5	
3	4	4	6	7	7	
5	5					
6	7					

$P \leftarrow j$:

1	1	1	2	**2**	3	5
2	2	2	3	**3**	5	
3	4	4	**4**	7	7	
5	5	**6**				
6	7					

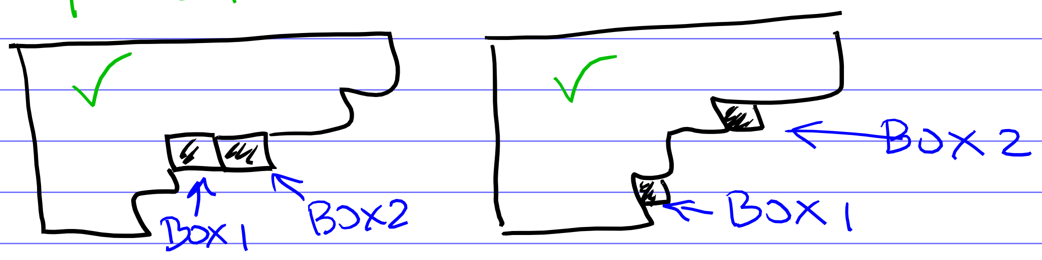
Lemma. If $j_1 \le j_2$

$P \leftarrow j_1 =$ 

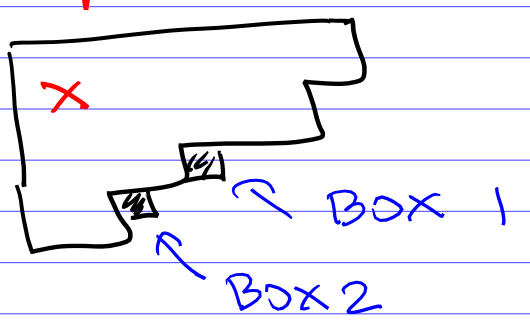
$(P \leftarrow j_1) \leftarrow j_2 =$ 

Then NEW BOX 2 is located immediately to the right, or above of NEW BOX 2

possible:

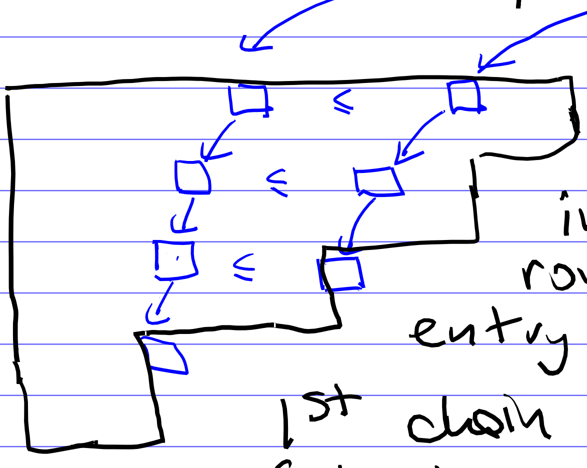


impossible:



Proof

$j_1 \le j_2$



in each row the entry in the 1st chain (starting with j_1) is strictly to the left and less than or equal to the entry in the 2nd chain (starting with j_2).

$\Rightarrow$ The second chain ends before or in the same row as the first chain. $\square$

RSK : $A=(a_{ij}) \rightsquigarrow (P, Q)$

pair of SSYTs of same shape λ

Convert A into biword

$$\binom{i_1}{j_1}\binom{i_2}{j_2} \cdots \binom{i_N}{j_N}$$

with a_{ij} pairs $\binom{i}{j}$ ordered lexicographically: $\binom{i}{j} < \binom{i'}{j'}$ if

$i < i'$ or $(i = i'$ & $j < j')$

insertion tableau:

$$P = ((\varnothing \leftarrow j_1) \leftarrow j_2) \leftarrow \cdots \leftarrow j_N$$

recording tableau:

Q : if at k-th insertion we added a box to P, the we add a box to Q located at the same position, filled with i_k.

Ex $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

biword $w = \binom{1}{2}\binom{2}{1}\binom{2}{1}\binom{2}{4}\binom{3}{3}$

$$P: \emptyset \overset{2}{\rightsquigarrow} \boxed{2} \overset{1}{\rightsquigarrow} \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \overset{1}{\rightsquigarrow} \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \overset{4}{\rightsquigarrow} \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} \overset{3}{\rightsquigarrow} \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

$$Q: \emptyset \rightsquigarrow \boxed{1} \rightsquigarrow \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$$

$$\underset{A}{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}} \xmapsto{RSK} \Bigg(\underset{P}{\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}} , \underset{Q}{\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}} \Bigg)$$

weights: $(2,1,1,1)$ $(1,3,1)$

column sums of A row sums of A

Clearly, $A \rightsquigarrow$ pair of SSYT's of the same shape

In order to show that this is a bijection, we need to construct inverse procedure.

Inverse Algorithm:

$(P, Q) \longmapsto \text{biword} \leadsto A$

(1) Find the horizontal strip in Q filled with largest entries of Q

(2) "Uninsert" entries of P tableau located in the same positions as in the strip of Q (found is (1)) in the order right to left

this follow from Lemma

(3) Repeat steps (1) & (2) untill the tableaux are empty.

Ex. $P = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$ $\quad Q = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}$

the largest entry in previous row which is < 4

$\left(\begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} , \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \right)$

need to uninsert entries in these positions from right to left

$\left(\begin{array}{|c|}\hline 2 \\ \hline\end{array} , \begin{array}{|c|}\hline 1 \\ \hline\end{array} \right)$

biword $\binom{1}{2} \binom{2}{1} \binom{2}{1} \binom{2}{4} \binom{3}{3}$

matrix $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

We proved

Theorem RSK is a bijection between matrices A & pairs (P, Q) of SSYT's of some shape λ s.t. column/row sums of A are weights of P & Q.

This proved Cauchy identity

$$\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_\lambda s_\lambda(x)\, s_\lambda(y)$$

How about dual Cauchy?

$$\prod_{i,j} (1+x_i y_j) = \sum_\lambda s_\lambda(x)\, s_{\lambda'}(y)$$

There is a similar construction, called the dual RSK that gives a bijection:

$$A \xleftrightarrow{\text{dual RSK}} (P, Q)$$

matrix filled with 0's & 1's

pairs of SSYTs of conjugate shapes

shape$(P) = \lambda$

shape$(Q) = \lambda'$

column/row sums of A = weights of P & Q.

RSK has many other important properties.

Theorem If $A \overset{RSK}{\longleftrightarrow} (P, Q)$ then $A^T \xrightarrow{RSK} (Q, P)$.

Can be proved. But not obvious from the classical construction of RSK.
We'll give another construction where this symmetry is manifest.

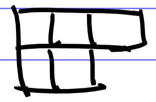
Theorem $A \overset{RSK}{\longmapsto} (P, Q)$ of shape λ.

Then λ_1 = the length of a "longest weakly increasing" subsequence of w

length of 1^{st} column of λ

λ'_1 = the length of a "longest strictly decreasing" subseq. of w

Ex. $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{RSK} (P, Q)$ of shape



$\lambda_1 = 3$

$\lambda_1' = 2$

$w = \begin{pmatrix} 1 & 2 & 2 & 2 & 3 \\ 2 & \boxed{1} & \boxed{1} & 4 & \boxed{3} \end{pmatrix}$

a longest weakly incr. subseq.

Note: Subsequence does not need to be consecutive

Problem. What is the length of a longest increasing subsequence in a permutation (distribution of the length, etc)

This was one of the reasons why Robinson-Schensted was initially introduced.

Let's talk about permutations, (Schensted's case)

We have $S_n \ni w \xleftrightarrow{\text{Schensted}} (P, Q)$

pair of SYTs of same shape λ with $|\lambda| = n$

$w = \begin{pmatrix} 1 & 2 & \dots & n \\ w_1 & w_2 & \dots & w_n \end{pmatrix}$ permutation

λ_1 : longest incr. subseq

$w_{i_1} < w_{i_2} < \dots < w_{i_{\lambda_1}}$

$i_1 < \dots < i_{\lambda_1}$

λ_1' : longest decreasing subseq.

$w_{j_1} > w_{j_2} > \dots > w_{j_{\lambda_1'}}$

$j_1 < j_2 < \dots < j_{\lambda_1'}$

<u>Symmetry</u> If $w \longleftrightarrow (P, Q)$

then $w^{-1} \longleftrightarrow (Q, P)$

Let $f_\lambda := \#\{\text{SYT's of shape } \lambda\}$

Later we'll give explicit formula for f_λ called the hook length formula

Corollary

(1) $$\sum_{\lambda \vdash n} f_\lambda^2 = n!$$

(2) $$\sum_{\lambda \vdash n} f_\lambda = \#\{\text{involutions in } S_n\}$$

$$= \#\{w \in S_n \mid w^{-1} = w\}$$

$$= \boxed{\sum_{k=0,\dots,\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (2k-1)!!}$$

Exercise (will be in P set 1)

where $(2k-1)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1)$.

Proof of (2)

$w \mapsto (P, Q)$, $w^{-1} \mapsto (Q, P)$

$P = Q$ iff $w = w^{-1}$, i.e. w is an involution in S_n

$\square$

Applications to Ramsey Theory

Corollary (Erdős-Szekeres Theorem)

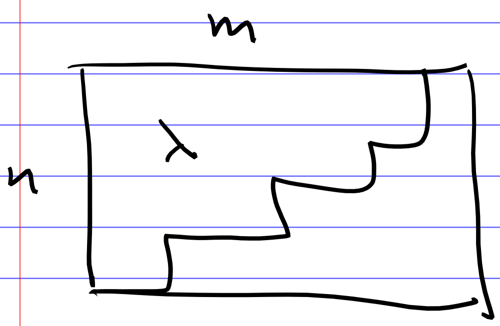
Fix $m, n \geq 1$. $N = m \cdot n + 1$.
Any permutation $w_1 w_2 \ldots w_N$ either has an increasing subsequence of length $m+1$ or a decreasing subseq. of length $n+1$

Proof. Suppose not.
longest incr. subseq. has length $\leq m$
& longest decr. subseq. —"— $\leq n$

$$w \xmapsto{\text{Schensted}} (P, Q) \text{ of shape } \lambda$$

$$\lambda_1 \leq m \quad \& \quad \lambda_1' \leq n$$

$\Downarrow$



λ fits inside $m \times n$ rectangle

$\Downarrow$

λ has $\leq m \cdot n$ boxes.

But we assumed that
$$N = |\lambda| = m \cdot n + 1. \qquad \square$$

There is another nice argument for Erdős - Szekeres Thm based on pigeonhole principle.

$\lambda_1 \Leftrightarrow$ increasing subseq.

How about $\lambda_2, \lambda_3, \ldots$?

Greene's Theorem

$w \overset{RSK}{\longmapsto} (P, Q)$ of shape λ

(a permutation or a biword)

Then $\forall\, k$

$$\lambda_1 + \ldots + \lambda_k = \max \# \left(\begin{array}{l}\text{union of} \\ k \text{ weakly} \\ \text{increasing} \\ \text{subseq. of } w\end{array}\right)$$

$$= \max_{\substack{\text{each } I_1, \ldots, I_k \\ \text{is increasing}}} \# (I_1 \cup \ldots \cup I_k).$$

$$\lambda_1' + \ldots + \lambda_k' = \max \# \left(\begin{array}{l}\text{union of} \\ k \text{ strictly} \\ \text{decr. subseq.}\end{array}\right)$$

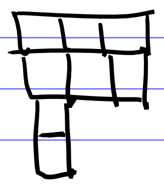
$$= \max_{\substack{\text{each } D_1, \ldots, D_k \\ \text{is decreasing}}} \# (D_1 \cup \ldots \cup D_k).$$

Example. $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 1 & 8 & 7 & 2 & 6 & 4 \end{pmatrix}$

shape of P & Q ?

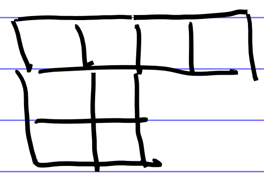
$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 1 & 8 & 7 & 2 & 6 & 4 \end{pmatrix}$

$$\left.\begin{array}{l} \lambda_1 = 3 \\ \lambda_1 + \lambda_2 = 6 \\ \lambda_1 + \lambda_2 + \lambda_3 = 7 \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 8 \end{array}\right\} \quad \lambda =$$



$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 1 & 8 & 7 & 2 & 6 & 4 \end{pmatrix}$

$$\left.\begin{array}{l} \lambda_1' = 4 \\ \lambda_1' + \lambda_2' = 6 \\ \lambda_1' + \lambda_2' + \lambda_3' = 8 \end{array}\right\} \quad \lambda' =$$



Warning: $\lambda_1 + \lambda_2 = \max \# (I_1 \cup I_2)$

I_1, I_2 are incr. subs.

but sometimes it is impossible to find such I_1 & I_2 s.t. I_1 is a max. incr. subsec.

How to prove these properties of RSK (symmetry, incr. & decr. subsequences, ...) ?

There is another construction of RSK where these properties become clear.

We want to break Schensted insertion steps into smaller, "more elementary" steps.

this leads to Fomin' growth diagrams (for SYTs

&

toggle operations (for SSYTs

Kirillov - Berenstein, & ....

Let's start with

permutations & SYT's.

$$S_n \xleftrightarrow{\text{Schensted}} \{(P,Q)\}$$

pair of SYTs of shape $\lambda \vdash n$.

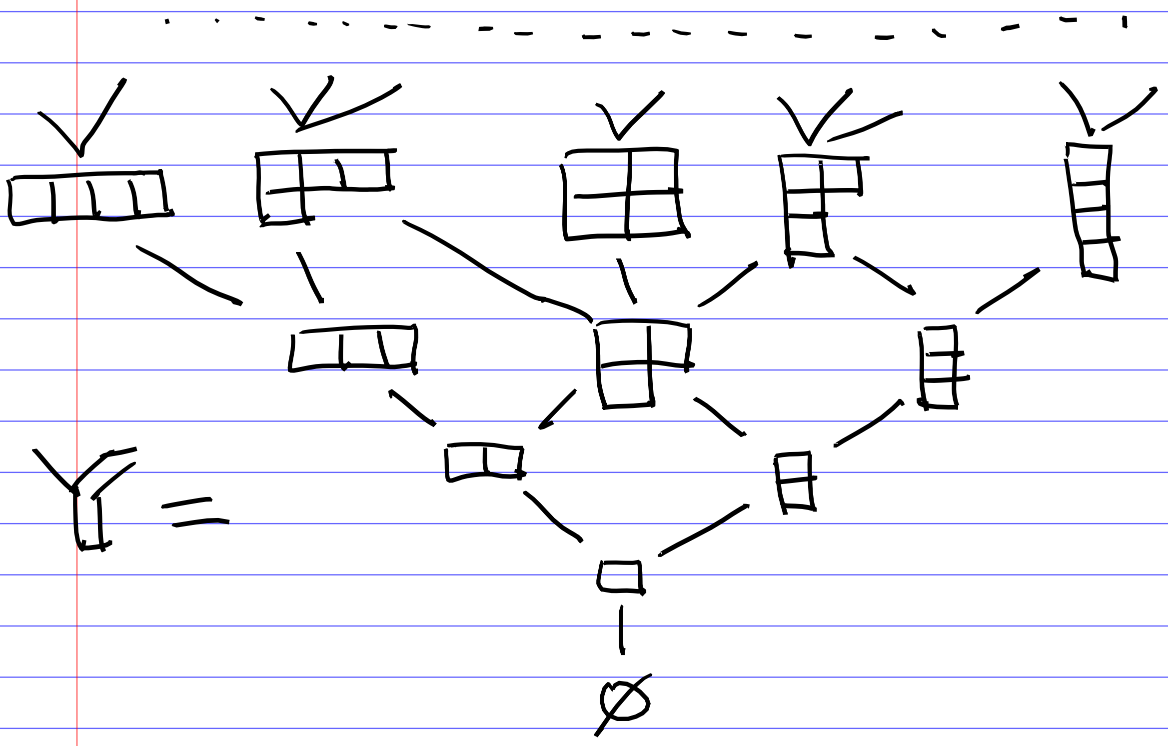
Young's lattice $\mathbb{Y}$

All Young diagrams ordered by inclusion $\lambda \le \mu$ if $\lambda \subseteq \mu$.

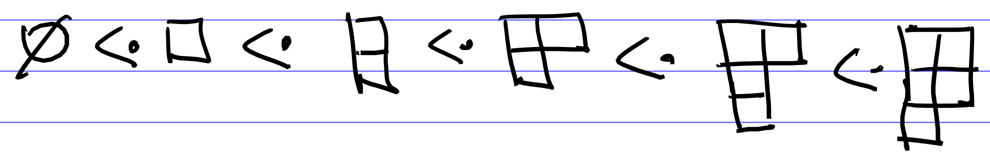
Covering relations in $\mathbb{Y}$

$\lambda \lessdot \mu$ if $\lambda \subseteq \mu$ & $|\mu/\lambda| = 1$

$\mathbb{Y} =$



An SYT corresponds to a saturated inreasing chain is $\mathbb{Y}$ from $\emptyset$ to λ

Ex. 

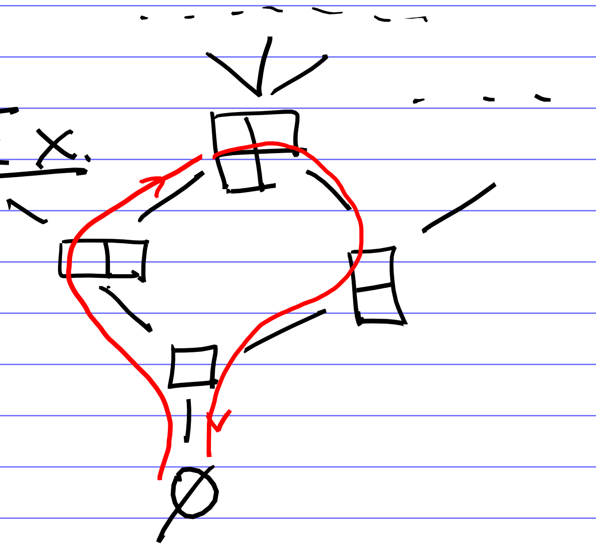
1	3
2	4
5	

A pair (P, Q) of SYT's of the same shape λ

corresponds to a path in $\mathbb{Y}$

$\emptyset \lessdot \square \lessdot \dots \lessdot \lambda \gtrdot \dots \gtrdot \emptyset$

from $\emptyset$ to $\emptyset$ with n "up" steps followed by n "DOWN" steps

Ex. 

$P =$

1	2
3	

$Q =$

1	3
2	

How about arbitrary paths in $\mathbb{Y}$ from $\varnothing$ to $\varnothing$?

Up & Down operators

acting on $\mathbb{Z}[\mathbb{Y}]$ (the space of formal linear combinations of Young diagrams)

$\mathbb{Z}[\mathbb{Y}]$ has linear basis given by λ's

Some elts of $\mathbb{Z}[\mathbb{Y}]$:

$\ydiagram{2}$, $\quad 2\,\square + 25\,\ydiagram{2,2} - \ydiagram{2,1,1}$

$3\cdot\varnothing + 100\cdot\square$, etc.

$U, D : \mathbb{Z}[\mathbb{Y}] \to \mathbb{Z}[\mathbb{Y}]$ two linear operators s.t.

$$U : \lambda \longmapsto \sum_{\mu : \mu \gtrdot \lambda} \mu$$

(μ obtained from λ by adding a single box)

$$D : \lambda \longmapsto \sum_{\mu : \mu \lessdot \lambda} \mu$$

μ is obtained from λ by removing a single box.

Ex. $U : \square \mapsto \ydiagram{2} + \ydiagram{1,1}$

$D : \square \mapsto \emptyset$

$$U^2(\emptyset) = U(\square) = \ydiagram{2} + \ydiagram{1,1}$$

$$U^3(\emptyset) = U\left(\ydiagram{2} + \ydiagram{1,1}\right) =$$

$$\ydiagram{3} + \ydiagram{2,1} + \ydiagram{2,1} + \ydiagram{1,1,1}$$

$$= \ydiagram{3} + 2\,\ydiagram{2,1} + \ydiagram{1,1,1}$$

etc

$$U^n(\emptyset) = \sum_{\lambda \vdash n} f_\lambda \, \lambda$$

$$D^3 U^3(\emptyset) = D^3\left(\ydiagram{3} + 2\,\ydiagram{2,1} + \ydiagram{1,1,1}\right)$$

$$= \emptyset + 2(2\cdot\emptyset) + \emptyset$$

$$= 6\,\emptyset.$$

How about

$$D^n U^n(\emptyset) = \,?$$

Theorem

$$D^n U^n(\emptyset) = n!\, \emptyset.$$

This is equivalent to

$$\sum_{\lambda \vdash n} f_\lambda^2 = n!$$

$\Updownarrow$

There are $n!$ of paths in $\mathbb{Y}$ from $\emptyset$ to $\emptyset$ with n "Up" steps followed by n "Down" steps.

We already know this from RSK.

But is there a simpler proof?

Lemma $DU - UD = I$

or $\boxed{[D, U] = I}$

Proof

$$\begin{tikzcd} & \mu \arrow[dr, "\text{down}"] & \\ \lambda \arrow[ur, "\text{up}"] \arrow[dr, "\text{down}"'] & & \nu \\ & \tilde{\mu} \arrow[ur, "\text{up}"'] & \end{tikzcd}$$

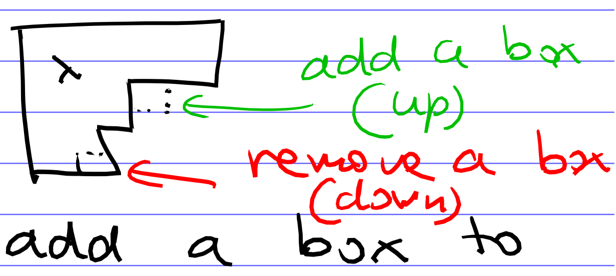
The coeff. of ν in

$(DU - UD)(\lambda) =$

$\#$ such μ's $-$ $\#$ such $\tilde{\mu}$'s

2 cases:

(I) $\lambda \neq \nu$



we want to add a box to λ & then remove a different box.

We can do these operations in a different order (first remove & then add)

So $[\nu](DU - UD)(\lambda) = 0$ if $\nu \neq \lambda$.

the coeff of ν

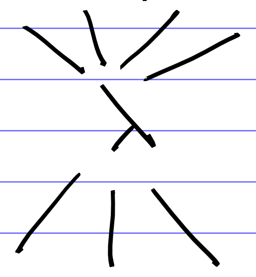
Case II $\lambda = \nu$

$\#\{$Add a box to λ & then remove the same box$\}$

$- \#\{$Remove a box from λ & then add the same box$\}$

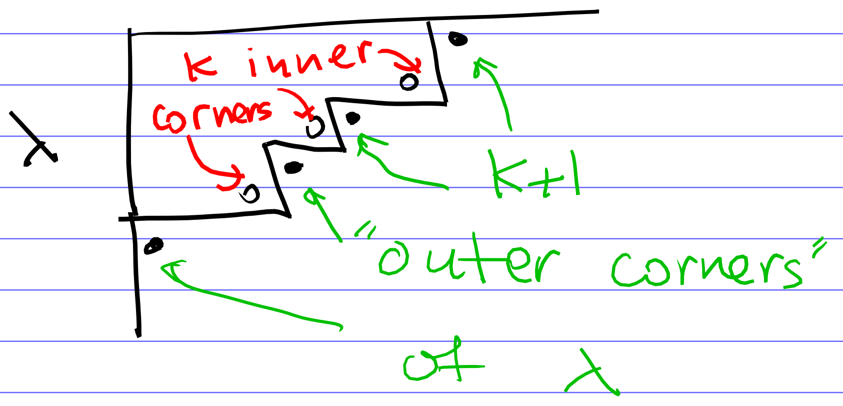
In II:

k+1 possible μ's



k possible $\tilde{\mu}$'s

Why?



So $[\lambda](DU - UD)(\lambda) =$

$(k+1) - k = 1$

We obtain $(DU - UD)(\lambda) = \lambda$.

Claim The relations

(1) $[D, U] = I$

(2) $D\varnothing = 0$

Imply that $D^n U^n(\varnothing) = n! \cdot \varnothing$

Ex $DDUU(\varnothing) =$

$= (DUDU + DU)(\varnothing)$

$= (DU\underbrace{UD}_{=0} + DU + \underbrace{UD}_{=0} + I)\varnothing$

$= (\underbrace{UD}_{=0} + I + I)(\varnothing) =$

$= 2I$

(each D is "moving" to the right until it "annihilates" with some U)

There are $n!$ ways to match all n D's with n U's into "annihilating pairs".

Another argument

U, D satisfy the same relations as the operators

$$X : f(x) \longmapsto x \cdot f(x)$$

$$\frac{d}{dx} : f(x) \longmapsto f'(x)$$

acting on polynomial ring $\mathbb{Z}[x]$ and $\frac{d}{dx}(1) = 0$.

So the calculation of $D^n U^n(\varnothing) = ?\, \varnothing$ is equivalent to

$$\left(\frac{d}{dx}\right)^n X^n(1) = \left(\frac{d}{dx}\right)^n (x^n)$$

$$= n! \cdot 1 \implies ? = n!$$

$\square$

This is why Stanley gave the following definition

Def. A poset P is called a *differential poset* if

(1) P has a unique minimal element $\hat{0}$.

(2) The "up" & "down" operators acting on $\mathbb{Z}[P]$ satisfy

$$[D\ U] = I.$$

Theorem. For any differential poset

$$D^n U^n(\hat{0}) = n!\,\hat{0}.$$