18.217 Lecture 4 09/11/2020

last time: 2 definitions of Schur polyn.

classical def. $$s_\lambda^{class}(x_1,\dots,x_n) := \frac{\sum_{w\in S_n} (-1)^{\ell(w)} w(x^{\lambda+\delta})}{\prod_{i<j}(x_i-x_j)}$$

$$\|?$$

combinatorial def. $$s_\lambda^{comb} := \sum_{T:SSYT(\lambda)} x^{weight(T)}$$

today we'll give 2 more constructions for s_λ.

But before, let's talk about the symmetric group S_n.

Def: S_n is the group of permutations, which are bijections $w: [n] \to [n]$.

$$[n] = \{1,\dots,n\}$$

Product of permutations u, w in S is composition of maps

$$u\circ w : i \longmapsto u(w(i)).$$

Simple transpositions $s_i = (i, i+1)$

$$s_i = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & \cdots & n \\ 1 & 2 & & i+1 & i & \cdots & n \end{pmatrix}$$

<u>Theorem</u>. S_n is generated by $s_1, s_2, \ldots, s_{n-1}$ with relations:

Coxeter relations

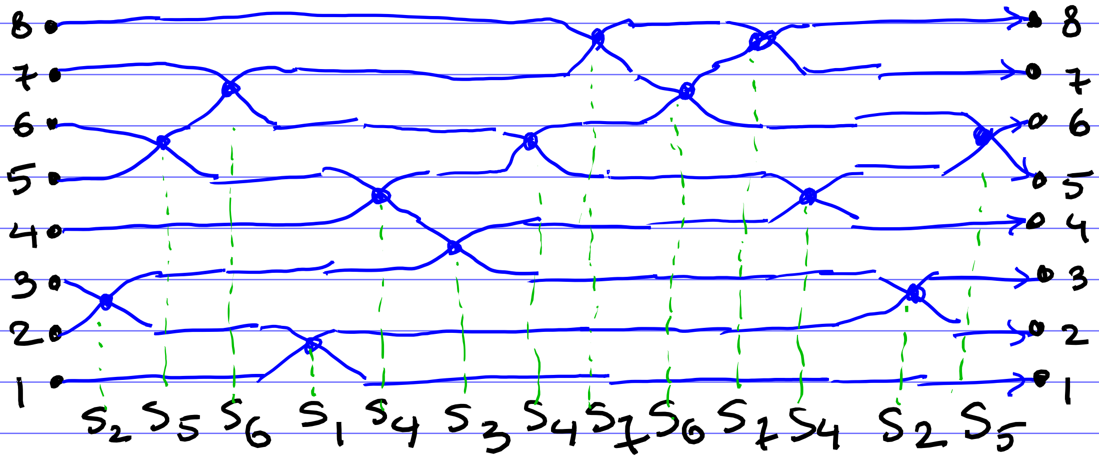
(1) $s_i^2 = 1$

(2) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

(3) $s_i s_j = s_j s_i, \quad |i-j| \geq 2,$

One can represent a permutation by its <u>wiring diagram</u>

<u>Example</u>. $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 1 & 8 & 7 & 2 & 4 & 5 \end{pmatrix}$



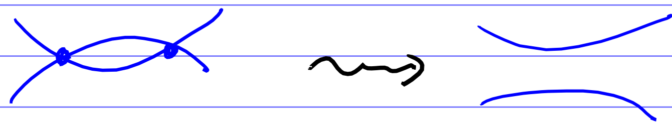
A wiring diagram represents a <u>decomposition</u> of w into a product of adjacent transpositions.

<u>above example</u>:

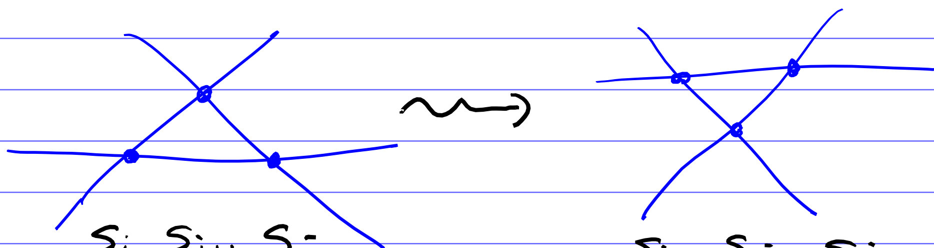
$$w = s_5 s_2 s_4 s_7 s_6 s_7 s_4 s_3 s_4 s_1 s_6 s_5 s_2.$$

written "backward" because compositions of functions "act backward".

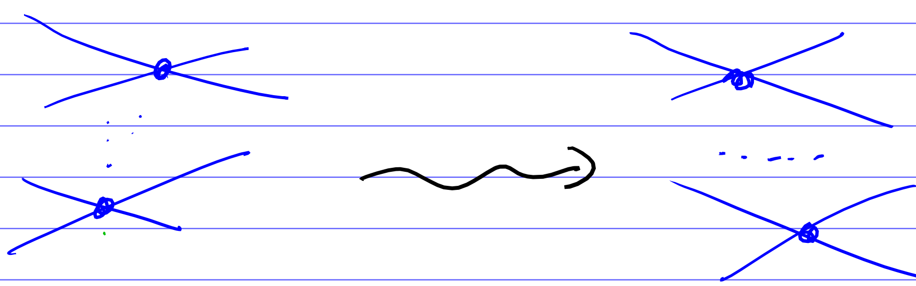
Any two wiring diagrams of the same permutation can be obtained from each other by a sequence of the following moves:

(1) 

$s_i \; s_i \qquad\qquad 1$

(2) 

$s_i \, s_{i+1} \, s_i \qquad\qquad s_{i+1} \, s_i \, s_{i+1}$

(3) 

$s_i \; s_j \qquad\qquad s_j \; s_i \qquad |i-j| \geq 2$

This explains why all relations in S_n are generated by the Coxeter relations (1), (2), (3).

Def.

the <u>length</u> $\ell(w)$ of $w \in S_n$

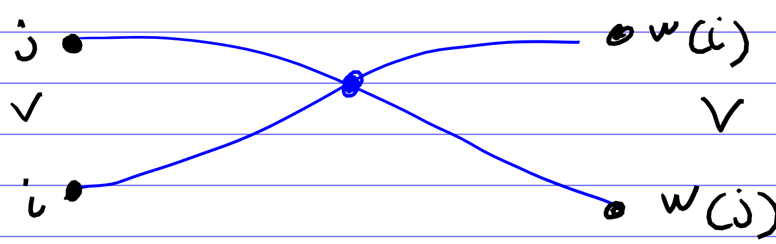
$:=$ the shortest length ℓ of a decomposition $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$.

$=$ minimal possible number of <u>crossings</u> in a wiring diagram for w

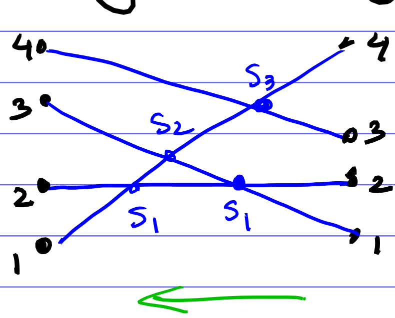
$=$ # <u>inversions</u> $\mathrm{inv}(w)$ of w

$\mathrm{inv}(w) := \# \{ i<j \mid w(i) > w(j) \}$.

Clearly, for an inversion we have (at least 1) crossing of wiries



We can construct a wiring diagr. with <u>exactly</u> $\mathrm{inv}(w)$ crossing by drawing straight wires.



$w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} =$

$= s_3 s_1 s_2 s_1.$

Def. A decomposition of w of minimal possible length $\ell = \ell(w)$ is called a <u>reduced decomposition</u>.

Ex. $w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} =$

$$= s_3 s_1 s_2 s_1 = s_3 s_1 s_2 s_3 s_3 s_1$$

reduced

not reduced

$$\| $$

$$s_3 s_1 s_2 s_3 s_1 s_3$$

Lemma*. Any two reduced decompositions for w can be obtained from each other by moves (2) & (3), <u>without</u> using the move (1)

$$s_i s_i \Longleftrightarrow 1.$$

<u>Problem</u> from 1^{st} Pset: Prove this.

The permutation group S_n <u>acts</u> on the polynomial ring

$$\mathbb{Z}[x_1, x_2, \ldots, x_n]$$

$$w : f(x_1, \ldots, x_n) \longmapsto f(x_{w(1)}, \ldots, x_{w(n)}).$$

We can write

$$w(f) := f(x_{w(1)}, \ldots, x_{w(n)}).$$

Now the classical def of s_λ can be written as

$$s_\lambda^{class}(x_1,\dots,x_n) := \frac{\sum_{w \in S_n} (-1)^{\ell(w)}\, w(x^{\lambda+\delta})}{\prod_{i<j}(x_i - x_j)}$$

Divided Differences

Def. Divided difference operators

$$\partial_i : \mathbb{Z}[x_1,\dots,x_n] \to \mathbb{Z}[x_1,\dots,x_n] \qquad i=1,\dots,n-1$$

$$\partial_i : f \longmapsto \frac{1}{x_i - x_{i+1}}(1-s_i)(f).$$

$$\Big\| $$

$$\frac{f(x_1\dots x_n) - f(x_1\dots x_{i+1}\, x_i \dots x_n)}{x_i - x_{i+1}}$$

Lemma. ∂_i is a valid operation on $\mathbb{Z}[x_1\dots x_n]$.

Proof. We have $\left(f - s_i(f)\right)\Big|_{x_i = x_{i+1}} = 0$

$\Rightarrow$ $f - s_i(f)$ is divisible by $x_i - x_{i+1}$. $\square$

Lemma. The operators $\partial_1, \dots, \partial_{n-1}$ satisfy the relations:

(1)' $\partial_i^2 = 0$

(2) $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$

(3) $\partial_i \partial_j = \partial_j \partial_i$, $|i-j| \geq 2$

Proof. Easy to verify by expanding all terms as $\partial_i = \frac{1}{x_i - x_{i+1}}(1 - s_i)$. $\square$

Def. $w \in S_n$

$w = s_{i_1} \dots s_{i_\ell}$ a reduced decomp.

$$\partial_w := \partial_{i_1} \dots \partial_{i_\ell}$$

By Lemma*, ∂_w depends only on w and not on a choice of reduced decomp.

Def $w_0 = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}$

the longest permutation in S_n

Schur as a Schubert polyn.

$$\boxed{S_\lambda^{\text{Schub.}} := \partial_{w_0}\left(x^{\lambda+\delta}\right)}$$

$\lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda \geq \dots \geq \lambda_n \geq 0$

$\delta = (n-1, n-2, \dots, 1, 0)$, $\quad x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$

<u>Theorem.</u> $S_\lambda^{\text{class.}} = S_\lambda^{\text{Schub.}}$

<u>Recall,</u> $$S_\lambda^{\text{class.}} := \frac{\left(\sum_{w \in S_n} (-1)^{\ell} w\right)\left(x^{\lambda+\delta}\right)}{\prod (x_i - x_j)}$$

Here we identify $w \in S_n$ with the operator $w : f \mapsto w(f)$ acting on $\mathbb{Z}[x_1, \dots x_n]$.

The above Thm follows from the identity.

Theorem. $\dfrac{1}{\prod_{1\le i<j\le n}(x_i-x_j)}\left(\sum_{w\in S_n}(-1)^{\ell(w)}w\right)$

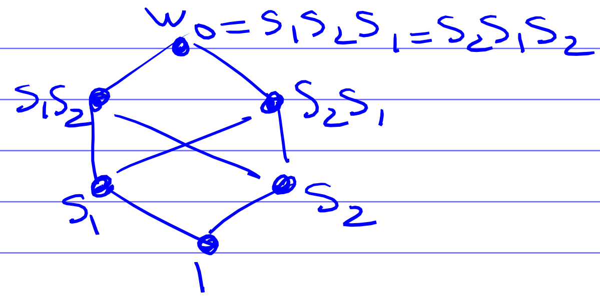
$= \partial_{w_0}$

∂_1

Example. $n=2$ $\quad \dfrac{1}{x_1-x_2}(1-s_1) = \dfrac{1}{x_1-x_2}(1-s_1)$

$n=3$

S_3 :



$$\frac{1}{(x_1-x_2)(x_1-x_3)(x_2-x_3)}\left(1-s_1-s_2+s_1s_2+s_2s_1 - s_1s_2s_1\right)$$

$$\overset{?}{=} \partial_1\partial_2\partial_1 =$$

$$= \frac{1}{x_1-x_2}(1-s_1)\,\frac{1}{(x_2-x_3)}(1-s_2)\,\frac{1}{(x_1-x_2)}(1-s_1).$$

Let's check the coeff of $s_1s_2s_1$ on both sides

LHS: $\dfrac{1}{(x_1-x_2)(x_1-x_3)(x_2-x_3)}(-s_1s_2s_1)$

RHS: $\dfrac{1}{x_1-x_2}(-s_1)\dfrac{1}{x_2-x_3}(-s_2)\dfrac{1}{x_1-x_2}(-s_1)$

$x_1 \qquad x_2 \quad x_3$

$= \dfrac{1}{x_1-x_2}\,\dfrac{1}{x_1-x_3}\,\dfrac{1}{x_2-x_3}(-s_1s_2s_1)$ ✓

Let's check the coeff of $Id \in S_3$

LHS: $\dfrac{1}{(x_1-x_2)(x_1-x_3)(x_2-x_3)}(Id)$

RHS: $\dfrac{1}{(x_1-x_2)}1\cdot\dfrac{1}{(x_2-x_3)}\cdot 1\cdot\dfrac{1}{(x_1-x_2)}1$

$$+\frac{1}{x_1-x_2}(-s_1)\frac{1}{(x_2-\underset{x_1}{x_3})}1\cdot\frac{1}{(\underset{x_2}{x_1}-\underset{x_1}{x_2})}(-s_1)$$

$$\frac{1}{(x_1-x_2)}\frac{1}{x_1-x_3}\cdot\frac{1}{x_2-x_1}\cdot 1 .$$

one can check that it equals LHS.

etc.

Here we are using:

$$w(f\cdot g) = w(f)\cdot w(g)$$

eg. $s_1\left(-\dfrac{1}{x_2-x_3}\cdot g\right) = -\dfrac{1}{x_1-x_3}s_1(g).$

<u>Problem</u> (from PSet 1). Prove the identity

$$\frac{1}{\prod_{i<j} x_i-x_j}\left(\sum_{w\in S_n}(-1)^{\ell} w\right) = \partial_{w_0}.$$

Another construction for S_λ.

The classical def. uses the anti-symmetrization of monomial $x^{\lambda+\delta}$.

Let's us now symmetrize a monomial.

$$x^a y^b \rightsquigarrow x^a y^b + x^{a-1} y^{b+1} + \dots$$

$a > b$ $$\dots + x^{b+1} y^{a-1} + x^b y^a$$

we are doing this in a saturated way: we are including all monomials between $x^a y^b$ & $x^b y^a$.

$$x^a y^b + x^{a-1} y^{b+1} + \dots + x^b y^a =$$

$$= x^a y^b \left(1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2 + \dots\right)$$

$$- x^{b+1} y^{a-1} \left(1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2 + \dots\right)$$

$$= \boxed{\frac{x^a y^b - \frac{y}{x} x^b y^a}{1 - \frac{y}{x}}}$$

this is called the isobaric divided difference

Def The Demazure operators (a.k.a. isobaric divided differences) as the operators $D_1, \dots, D_{n-1}$ acting on $\mathbb{Z}[x_1, \dots, x_n]$ as

$$\boxed{D_i : f \longmapsto \frac{f - \frac{x_{i+1}}{x_i} s_i(f)}{1 - \frac{x_{i+1}}{x_i}}}$$

Lemma. D_i are valid operators on $\mathbb{Z}[x_1, \dots, x_n]$.

Proof $D_i(f) = \dfrac{x_i f - s_i(x_i f)}{x_i - x_{i+1}}$

$$= \partial_i(x_i f) \quad \square$$

Remark. Demazure operators & divided differences can be generalize to any Weyl group W.
[In this class, we'll consetrate mostly on type A case $W = S_n$]
Many things that we are doing here can be extended to any W.
But the relation
$D_i(f) = \partial_i(x_i f)$ is special to type A.

Clearly

$\partial_i : f \mapsto \partial_i(f)$ of degree $= \deg(f)-1$

$D_i : f \mapsto D_i(f)$ same degree as f.

Lemma $D_1, \ldots, D_{n-1}$ satisfy:

(1)" $D_i^2 = D_i$

(2) $D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1}$

(3) $D_i D_j = D_j D_i$, $|i-j| \geq 2$.

So, as before, we can define

Def For $w \in S_n$

$D_w := D_{i_1} \ldots D_{i_\ell}$ for

any reduced decomposition

$w = s_{i_1} s_{i_2} \ldots s_{i_\ell}$.

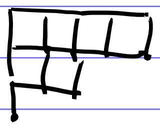
Schur polyn. as a Demazure character.

$$\boxed{s_\lambda^{\text{Dem.}} := D_{w_0}(x^\lambda)}$$

$\lambda = (\lambda_1, \dots, \lambda_n)$, $\quad x^\lambda = x_1^{\lambda_1} \dots x_n^{\lambda_n}$

Theorem.

$$s_\lambda^{\text{class.}} = s_\lambda^{\text{comb.}} = s_\lambda^{\text{Schub.}} = s_\lambda^{\text{Dem.}}$$

Example $n=3$, $\lambda = (4,2,0) =$ 

Let's calculate $s_{(4,2,0)}(x_1, x_2, x_3)$

using Demazure operators:

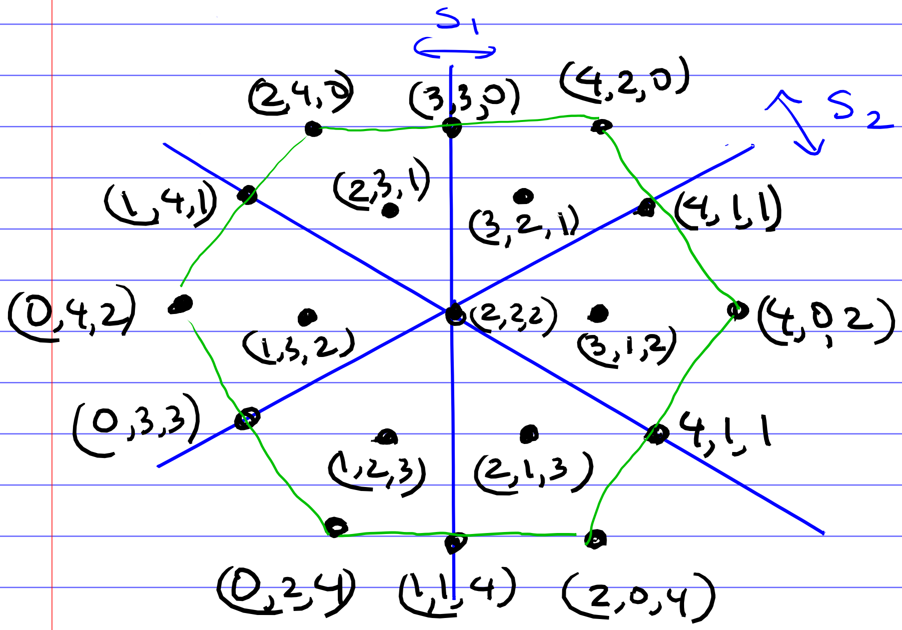
$$s_\lambda(x_1, x_2 x_3) = D_1 D_2 D_1 (x^\lambda)$$

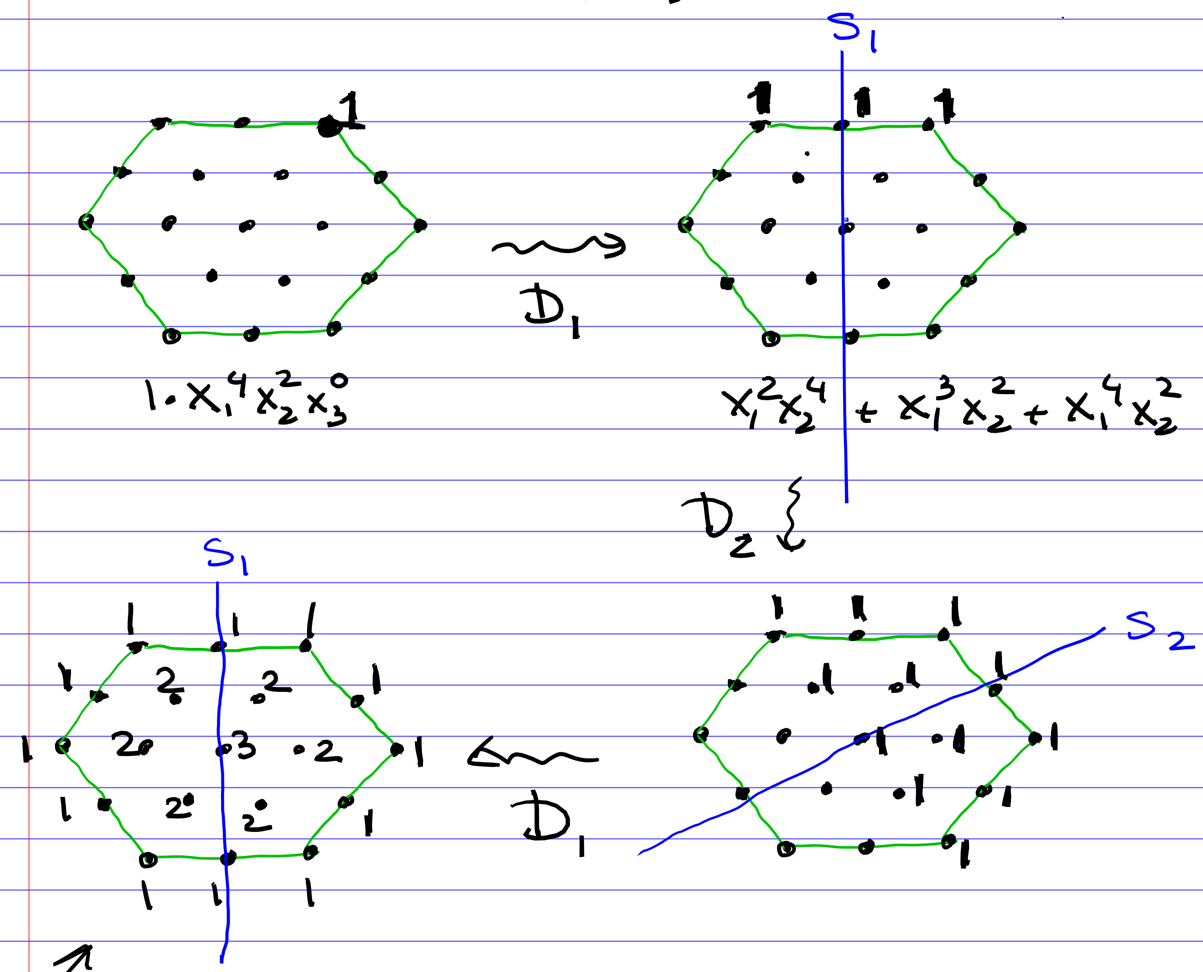
We'll represent a monomial $x_1^a x_2^b x_3^c$ by a point (a, b, c)

in the affine plane

$\{(x, y, z) \mid x+y+z = 6\} \subset \mathbb{R}^3$

All monomials will have same degree $= 6$, so we are "living" in an affine plane $x+y+z = 6$





$$S_{(4,2,0)} = x_1^2 x_2^4 + x_1^3 x_2^3 + x_1^4 x_2^4 +$$

$$+ x_1 x_2^4 x_3 + 2 x_1^2 x_2^3 x_3 + 2 x_1^2 x_2^3 x_3 + x_1^4 x_2 x_3$$

$$+ x_2^4 x_3^2 + 2 x_1 x_2^3 x_3^2 + 3 x_1^2 x_2^2 x_3^3 + 2 x_1^3 x_2 x_1^2 + x_1^4 x_3$$

. . . . . .

1 2 2 1

1 1 1

Problem Prove that

$$S_\lambda(x_1 \dots x_n) = D_{w_0}(x^\lambda).$$