18.217 Lecture 37 12/07/2020

Generalized parking functions (cont'd)

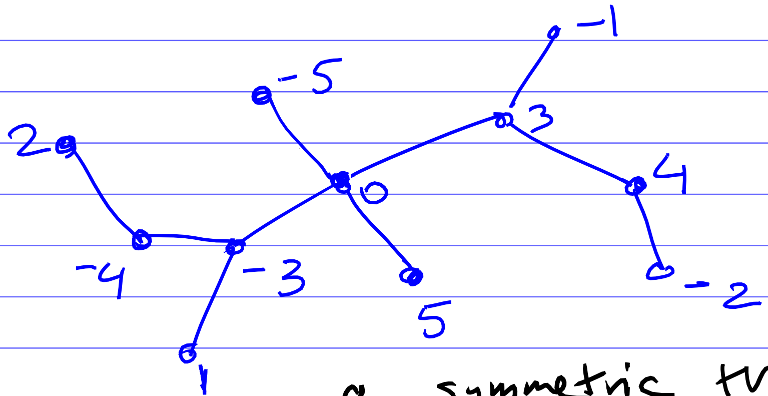
$\gamma = (\gamma_1 \geq \dots \geq \gamma_n)$ a partition w/ n parts.

Def. $f = (f_1, \dots, f_n)$, $f_i \in \mathbb{Z}_{\geq 0}$, is a γ-parking function if its decreasing rearrangement $g_1 \geq g_2 \geq \dots \geq g_n$ satisfies $g_i \leq \gamma_i$ $\forall\, i = 1, \dots, n$.

Example. $\gamma = (2n-1, 2n-3, 2n-5, \dots, 1)$

Claim ① # γ-parking functions $= (2n+1)^{n-1}$

② They are in bijection with trees T on $2n+1$ vertices labelled by $-n, -n+1, \dots, -1, 0, 1, 2, \dots, n$ s.t. $\varphi(T) = T$ where φ is induced by the map $i \leftrightarrow -i$, for $i = 0, \dots, n$

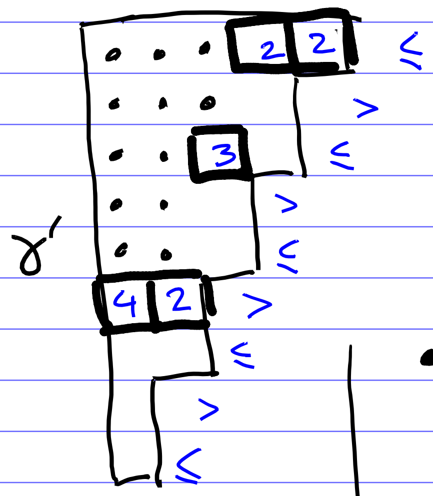


a symmetric tree

We defined a graded representation of S_n on γ-parking functions and corresponding symmetric functions (Frobenius char.)

$$pf_{\gamma,k} := \sum_{\substack{T \text{ semi-standard} \\ \gamma\text{-parking-function} \\ \text{tableau}}} x^T$$

<u>Example</u> $\gamma = (9,7,5,3,1)$, $k = 12$



a γ-p.f. tableau T

- shape of T is a horizontal n-strip that fits inside γ'
- entries weakly increase in rows $1, 3, 5, \ldots$
- entries strictly decrease in rows $2, 4, 6, \ldots$
- $k = \#$ boxes of γ' above T.

<u>Theorem</u> $(-1)^{|\gamma|-n} \sum_{k \geq 0} (-1)^k \, pf_{\gamma,k}$

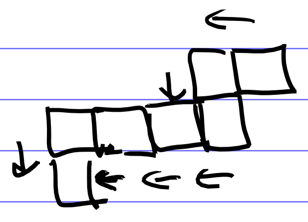
$$= \begin{cases} 0 & \text{if } \gamma_n \text{ is even} \\ s_\varkappa & \text{if } \gamma_n \text{ is odd}, \end{cases}$$

where $\varkappa$ is the ribbon associated with γ:

<u>Example</u> $\gamma = (8,7,7,7,6,3,3)$

e o o o e o o

$\downarrow \leftarrow \leftarrow \leftarrow \downarrow \leftarrow =$

$\varkappa =$ 

In particular

$$\sum_{\substack{(f_1,\ldots,f_n) \\ \gamma\text{-parking} \\ \text{function}}} (-1)^{|\gamma| - (f_1 + \cdots + f_n)} = \# \text{ SYT's of ribbon shape } \varkappa.$$

__Proof__ (Involution Principle)

Let's construct a sign reversing involution on γ-p.f. tableaux T.

Let σ_i be the entry of T in the column i.

Assign the direction $\varepsilon_i \in \{1,-1\}$ ($\varepsilon_i = 1$ is "down", $\varepsilon_i = -1$ is "up") for each entry σ_i, as follows:

$$\varepsilon_i = \mathrm{sgn}(\sigma_i - \sigma_{i+1})\,(-1)^{\mathrm{row}(\sigma_i)}$$

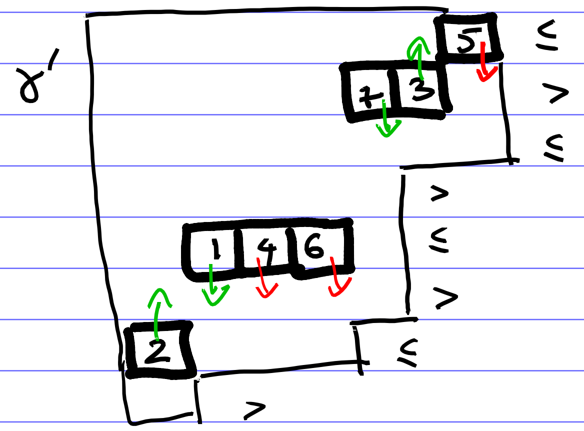
the index of row containing σ_i

$$= \begin{cases} 1 & \text{if } \sigma_i > \sigma_{i+1} \\ -1 & \text{if } \sigma_i \leq \sigma_{i+1} \end{cases}$$

(assuming $\sigma_{n+1} = n+1$).

We say that a box of T is movable (up or down) if moving it in the assigned direction would produce a valid γ-parking-function tableau.

Example $n=7$, $\gamma=(8,7,7,7,6,3,3)$



movable boxes in a γ-p.f. tableau.

Involution: If T has at least one movable box, then move the rightmost movable box in the assigned direction.

Then

- this map is an involution on the set of tableaux with at least 1 movable box
- It changes the parity of k $(= \#$ boxes above $T)$

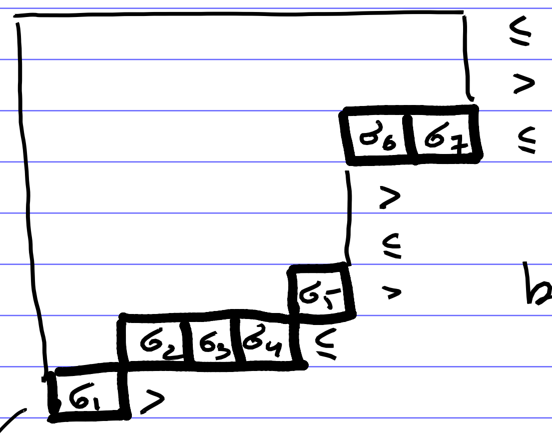
So this involution cancels all terms in $(-1)^{|\gamma|-n} \sum_k (-1)^k \, pf_{\gamma,k}$ corresponding to tableaux with at least 1 movable box

Lemma If γ_n is even, then $\exists$ a movable box

So $\sum (-1)^k \, pf_{\gamma,k} = 0$.

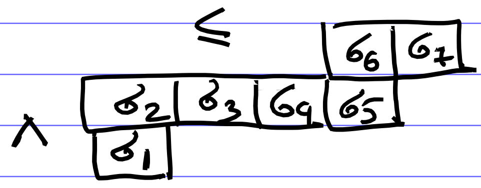
Lemma If γ_n is odd then γ-p.f. tableaux w/o movable boxes correspond to semi-standard Young tableaux of ribbon shape γ.

Ex $\gamma = (8,7,7,7,6,3,3)$



$\leftarrow$ entries are in the bottom boxes of columns of γ

$\sigma_1 > \sigma_2 \le \sigma_3 \le \sigma_4 \le \sigma_5 > \sigma_6 \le \sigma_7$



a SSYT of ribbon shape γ

So $(-1)^{|\gamma|-n} \sum_{k \ge 0} (-1)^k \, pf_{\gamma,k}$

$= S_\gamma$ $\square$

This construction is an example of how some classical combinatorial objects (parking functions, alternating permutations, ..) can be extended to symmetric functions)

Here is another (more well known) example.

Stanley's chromatic symmetric function

$G = (V, E)$ simple graph

A <u>proper</u> <u>coloring</u> of G

a function $c : V \to \{1, 2, \ldots\}$

s.t. $c(u) \neq c(v)$ for any edge $(u, v) \in E$.

<u>Def</u> A proper <u>k-coloring</u> is a proper coloring $c : V \to \{1, 2, \ldots, k\}$

<u>Lemma</u>. There exists a unique polynomial $\chi_G(t)$ such that, for any $k \in \mathbb{Z}_{>0}$, $\chi_G(k)$ is the number of proper k colorings of graph G.

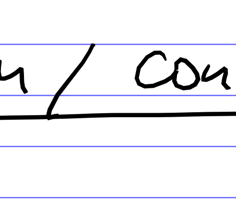
<u>Def</u> $\chi_G(t)$ is called the <u>chromatic polynomial</u> of G.

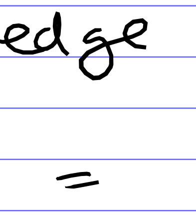
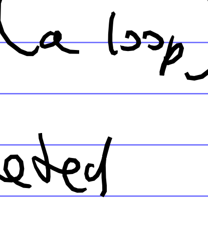
It is not hard to prove this Lemma (polynomiality of # proper k-colorings) by induction using <u>deletion / contraction</u>

e any edge of G (not a loop)

$G \setminus e$ = G with deleted edge e

G / e = G with contracted edge e

<u>Ex</u> $G =$ 

$G \setminus e =$  , $G/e =$ 

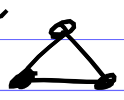
Then it is easy to see

$$\#\left\{\begin{matrix}\text{proper}\\ k\text{-colorings}\\ \text{of } G\end{matrix}\right\} = \#\left\{\begin{matrix}\text{proper}\\ k \text{ colorings}\\ \text{of } G \setminus e\end{matrix}\right\} - \#\left\{\begin{matrix}\text{proper}\\ k \text{ colorings}\\ \text{of } G/e\end{matrix}\right\}$$

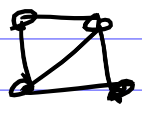
<u>Base of induction</u>:

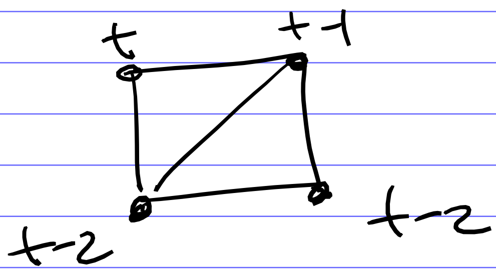
$$\#\left\{\begin{matrix}\text{proper } k \text{ colorings}\\ \text{of the empty}\\ \text{graph with } n\\ \text{vertices}\end{matrix}\right\} = k^n$$

Examples

χ  $(t) = t(t-1)(t-2)$

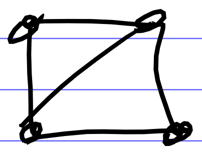
$\chi_{K_n}(t) = t(t-1)(t-2)\cdots(t-n+1)$

χ  $= t\cdot(t-1)(t-2)(t-2)$



Theorem (R. Stanley, 1973)

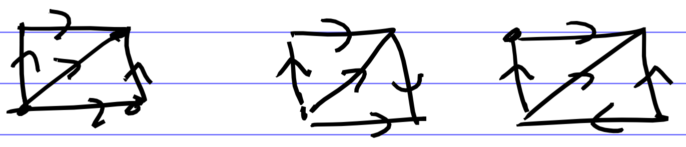
$$(-1)^{|V|}\chi_G(-1) = \#\left\{\begin{array}{l}\text{acyclic}\\ \text{orientations}\\ \text{of } G\end{array}\right\}$$

Example $G =$ 

$$\chi_G(t) = t(t-1)(t-2)^2$$

$$(-1)^{|V|}\chi_G(-1) = 1\cdot 2\cdot 3^2 = 18.$$

18 acyclic orientations of G



...

all ways to orient edges of G s.t. we don't create directed cycles.

Also not hard to prove this theorem using deletion/contraction and the lemma.

Lemma. $AO(G) = AO(G\setminus e) + AO(G/e)$

#acyclic orientations of G.

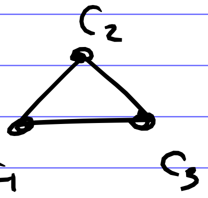
This is the classical story...

Can we lift to the level of symmetric functions?

Def Stanley's chromatic symmetric function of G.

$$X_G = X_G(x_1, x_2, \dots) :=$$

$$:= \sum_{\substack{c : V \to \{1,2,\dots\} \\ \text{proper colorings of } G}} \prod_{v \in V} x_{c(v)}$$

Example $G = K_3 =$ 

$c_i \neq c_j$

$$X_{K_3} = \sum_{\substack{c_1, c_2, c_3 \in \mathbb{Z}_{>0} \\ c_i \neq c_j}} x_{c_1} x_{c_2} x_{c_3}$$

$$= 6 \sum_{c_1 < c_2 < c_3} x_{c_1} x_{c_2} x_{c_3} =$$

$$= 6 \, e_3(x_1, x_2, \dots)$$

Similarly,

$$X_{K_n} = n! \, e_n$$

A Relationship between

$\chi_G(t)$ & X_G ?

chromatic polynomial

chromatic symmetric function

$\chi_G(k)$ = the specialization $X_G(\underbrace{1,1,\dots 1}_{k},0,0\dots)$

How to extend Stanley's theorem about acyclic orientations to X_G ?

By the Fundamental Thm on symmetric functions

$$X_G = \sum_{\lambda \vdash n} c_\lambda e_\lambda$$

$(n = |V|)$

some coefficients

Observation

$$e_r(\underbrace{1,\dots,1}_{k},0\dots0) = \binom{k}{r}$$

$$= \frac{1}{r!}\, k\cdot(k-1)\dots(k-r+1)$$

a polynomial in k.

Its value at $k=-1$ is

$$\binom{-1}{r} = \frac{1}{r!}(-1)(-2)\dots(-r) =$$

$$= (-1)^r.$$

Thus the specialization of $e_\lambda(\underbrace{1\dots1}_{k},0\dots0)$ at $k=-1$ is $(-1)^{|\lambda|}$.

Apply this to

$$X_G = \sum_\lambda c_\lambda e_\lambda$$

specialize at $(x_1,x_2,\dots) = (\underbrace{1,\dots,1}_{k},0,0,\dots)$

and then take $k=-1$

$$AO(G) = \sum_\lambda c_\lambda$$

So the coefficients c_λ "refine" the number $AO(G)$ of acyclic orientations of G.

Def. Let $AO_m(G) :=$

$:=$ # acyclic orientations of G with exactly m sinks.

The sink Theorem (also due to Stanley, 1995)

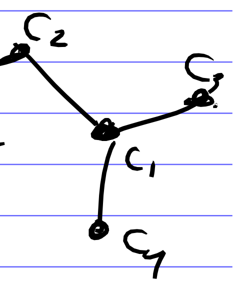
Let $X_G = \sum_{\lambda \vdash n} c_\lambda e_\lambda$

$(n = \# V)$.

Then $\sum'_{\lambda \vdash n} c_\lambda = AO_m(G)$.

partitions with exactly m (non-zero) parts

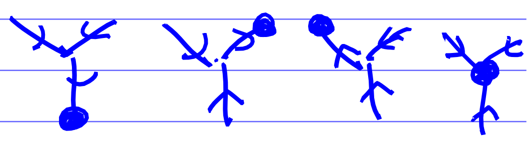
Example $G = K_{13} =$



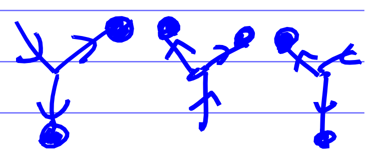
$$X_{K_{13}} = \sum_{\substack{c_1, c_2, c_3, c_4 \in \mathbb{Z}_{>0} \\ c_1 \neq c_2,\ c_1 \neq c_3,\ c_1 \neq c_4}} x_{c_1} x_{c_2} x_{c_3} x_{c_4}$$

$$= 4 e_4 + 5 e_{31} - 2 e_{22} + e_{211}$$

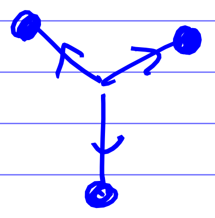
$AO_1(K_{13}) = 4$



$AO_2(K_{13}) = 5 - 2 = 3$



$AO_3(K_{13}) = 1$



Q: when are all the coefficients c_λ nonnegative

(In this case, we'll say the X_G is e-positive)

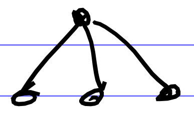
i.e. it is a linear combination of some e_λ's with positive coefficients.

A weaker property:

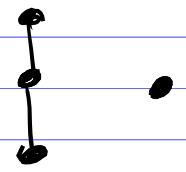
A symmetric function is called Schur-positive (or s-positive) if it is a linear combination of some Schur functions with positive coefficients.

Some conjectures on e-positivity & Schur-positivity of X_G.

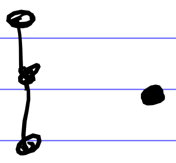
Def. A graph G is called claw-free if it does not have an induced subgraph isomorphic to

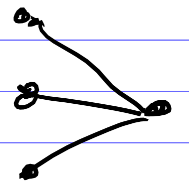
K_{13} =  (claw)

Def A poset P is called $(3+1)$-free if it does not contain an induced subposet isomorphic to



For a poset P on the set of elements V, its incomparability graph is the graph $G=(V,E)$ such that $(u,v) \in E$ if elements u & v are incomparable in P.

Example $P=$ 

the incomparability graph = 

Lemma If G is the inc. graph of P then

P is $(3+1)$-free iff G is claw-free

<u>Conjecture</u> (Stanley 1995, Stanley-Stanbridge 1993)

X_G is e-positive if

G is the incomparability graph of a (3+1)-free poset.

<u>Theorem</u> (Gasharov, 1996) If G is the incomparability graph of a (3+1)-free poset then X_G is Schur positive.

<u>Conjecture</u> (Gasharov) X_G is Schur-positive if G is claw-free.

<u>Remark</u>. Claw-free graphs form a more general class of graphs than incomparability graph of (3+1)-free posets. For such graphs X_G may not be e-positive.

Example. For $n \geq k \geq 1$, let G the graph on the vertex set $V = \{1, 2, \ldots, n\}$ with the edge set $E = \{(i,j) \mid |i-j| \leq k-1\}$.

Its chromatic symm. function is

$$P_{n,k} := \sum_{\substack{i_1, \ldots, i_n \in \mathbb{Z}_{>0} \text{ s.t.} \\ \text{any } k \text{ consecutive} \\ \text{terms are distinct}}} x_{i_1} \cdots x_{i_n}$$

Conjecture $P_{n,k}$ is e-positive.

The case $k = 3$ follows from the identity due to Carlitz

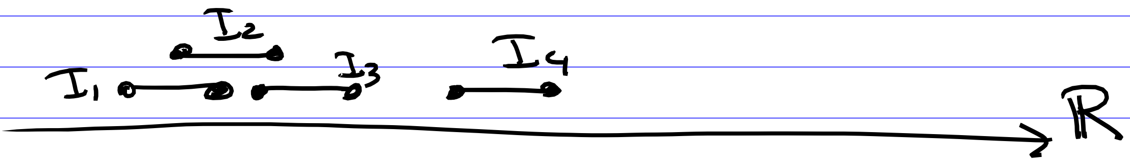
$$\sum_{n \geq 0} P_{n,2}\, t^n = \frac{\sum_{i \geq 0} e_i t^i}{1 - \sum_{i \geq 1} (i-1) e_i t^i}.$$

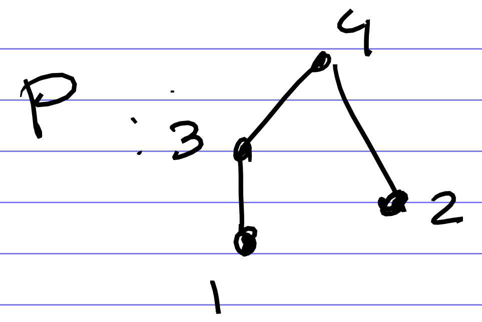
$\Rightarrow P_{n,2}$ is e-positive.

... more on (3+1)-free posets.

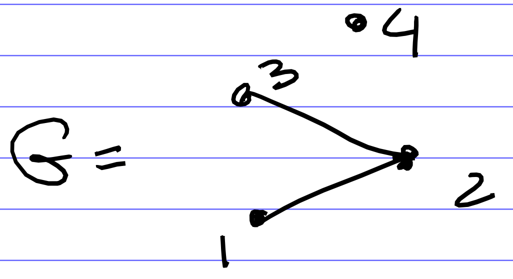
Def. A poset P is called a *unit interval order* (aka *semiorder*) if we can find a collection of unit intervals $I_1, \dots, I_n$ on the line $\mathbb{R}$ (associated with elements of P) s.t. $i <_P j$ iff the interval I_i lies to the left of I_j

Example.



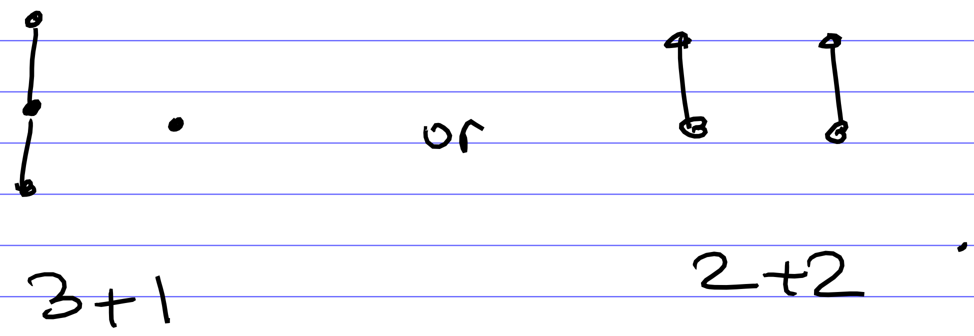


the incomparability graph of P



i & j are connected by an edge in G iff the intervals I_i & I_j overlap : $I_i \cap I_j \neq \emptyset$.

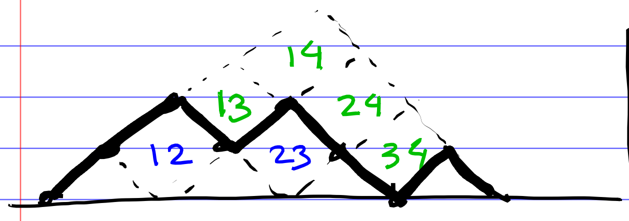
Theorem (Scott, Suppes 1958) A poset P is a unit interval order iff it is (3+1)-avoiding & (2+2)-avoiding i.e. it does not contain induced subposets of the form:



The number of (unlabelled) unit interval orders on n vertices equals the Catalan number C_n.

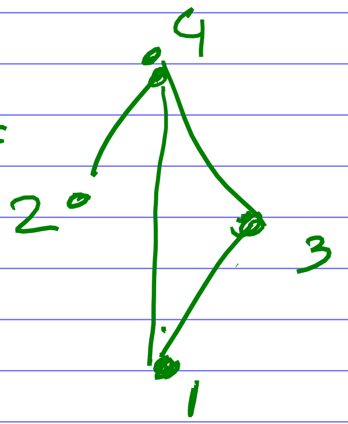
The bijection with Dyck paths:

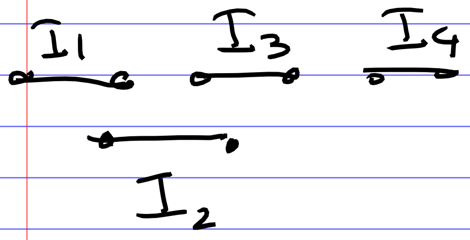
Example: $n = 4$



a Dyck path

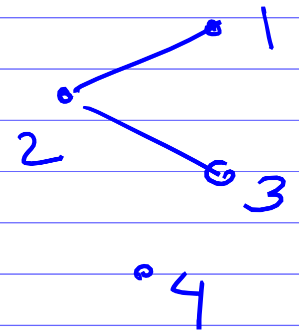
$P =$





$i <_P j$ if box (i,j) is above the Dyck path

$G =$



edges (i,j) the incomp. graph G correspond to boxes (i,j) below the Dyck path.

Any Dyck path $\rightsquigarrow$ graph G

$\rightsquigarrow$ the chromatic symm. function X_G.

According to Stanley's conj. such X_G should be e-positive.