18.217 Lecture 36 12/04/2020

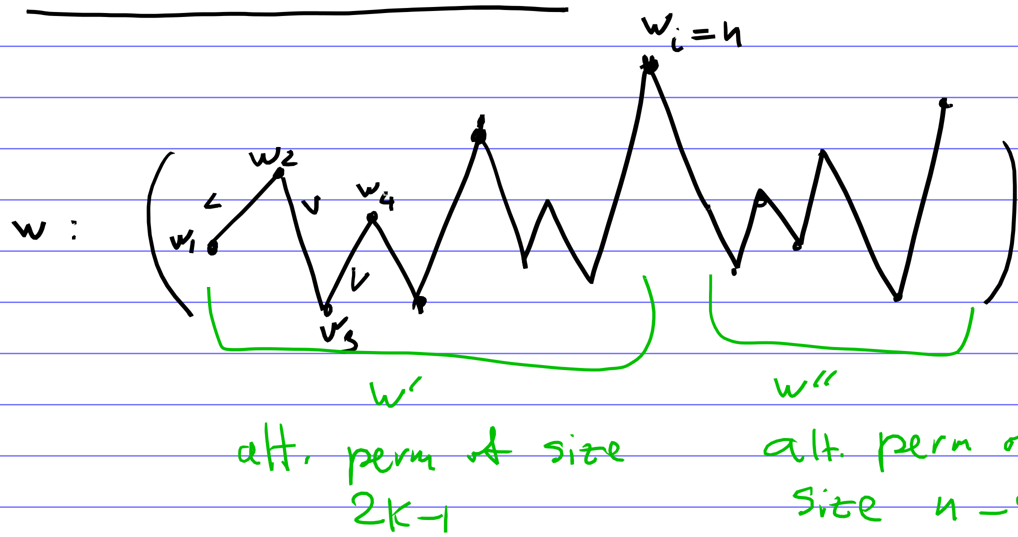
The Euler (zigzag) numbers: alternating permutation

$$A_n := \#\{w \in S_n \mid w_1 < w_2 > w_3 < \dots\}$$

(The Euler numbers should not be confused with the Eulerian numbers. The later are defined as #'s of permutations in S_n with a given number k of descents.)

Recurrence Relation

i should be even $i=2k$



$$(*) \quad \boxed{A_n = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1}{2k-1} A_{2k-1} \cdot A_{n-2k}, \quad \text{for } n \geq 2}$$

$$A_0 = A_1 = 1$$

Exponential generating function:

$$A(x) := \sum_{n \geq 0} A_n \frac{x^n}{n!}$$

$$A(x) = A^{even}(x) + A^{odd}(x)$$

$$A^{even}(x) := \sum_{n \in \{0,2,4,\dots\}} A_n \frac{x^n}{n!}$$

$$A^{odd}(x) := \sum_{n \in \{1,3,5,\dots\}} A_n \frac{x^n}{n!}$$

Rec. Relation $(*)$ $\Longleftrightarrow$

$$A_n \frac{x^{n-1}}{(n-1)!} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} A_{2k-1} \frac{x^{2k-1}}{(2k-1)!} \cdot A_{n-2k} \frac{x^{n-2k}}{(n-2k)!}$$

$(*)$ $\Longleftrightarrow$ differential equation for $A(x)$:

$$\boxed{\begin{aligned} A'(x) &= A^{odd}(x) \cdot A(x) + 1 \\ A(0) &= 1 \end{aligned}}$$

or
$$\begin{cases} \left(A^{odd}(x)\right)' = A^{odd}(x)\, A^{odd}(x) + 1 \\ A^{odd}(0) = 0 \end{cases}$$

$$\begin{cases} \left(A^{even}(x)\right)' = A^{odd}(x) \cdot A^{even}(x) \\ A^{even}(0) = 1 \end{cases}$$

Theorem $A(x) = \tan(x) + \sec(x)$

$$A^{odd}(x) = \tan(x), \quad A^{even}(x) = \sec(x)$$

Proof. Check that the same diff. eqns hold for $\tan(x)$ & $\sec(x)$

$$\tan(x)' = \sec(x)^2 + 1, \quad \tan(0) = 0$$
$$\sec(x)' = \sec(x) \cdot \tan(x), \quad \sec(0) = 1$$

The Euler-Bernoulli triangle
(or Entringer triangle)

$$
\begin{array}{ccccccccccc}
 & & & & & \boxed{1} = A_0 & & & & & \\
 & & & & 0 & & 1 = A_1 & & & & \\
 & & & A_2 = \boxed{1} & & 1 & & 0 & & & \\
 & & 0 & & 1 & & 2 & & 2 = A_3 & & \\
 & A_4 = \boxed{5} & & 5 & & 4 & & 2 & & 0 & \\
0 & & 5 & & 10 & & 14 & & 16 & & 16 = A_5
\end{array}
$$

. . . - - - - ,

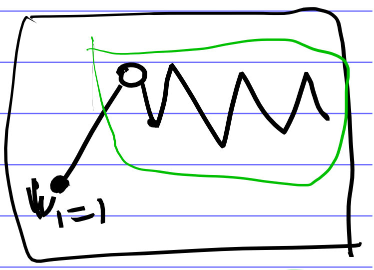
The Entringer numbers

w is alternating

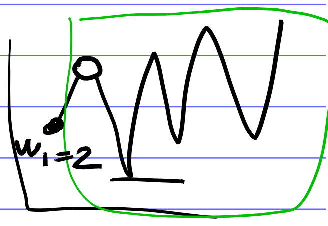
$$E_{nk} := \# \{ w \in S_n \mid w_1 < w_2 > w_3 < \dots, \; w_1 = k+1 \}$$

$$A_n = \sum_k E_{nk}$$

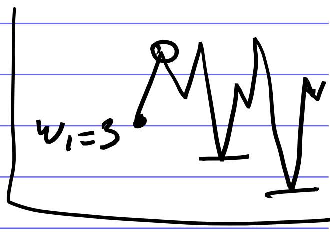
$$E_{n0} = \sum_{k=0}^{n-1} E_{n-1,k}$$



$$E_{n1} = \sum_{k=0}^{n-2} E_{n-1,k}$$



$$E_{n2} = \sum_{k=0}^{n-3} E_{n-1,k}$$



etc

This is the rec. relation in the above Euler-Bernoulli triangle.

Parking functions

Def. $f = (f_1, \ldots, f_n)$, $f_i \in [n]$, is a *parking function* if its decreasing rearrangement $g_1 \geq g_2 \geq \ldots \geq g_n$ satisfies:

$g_1 \leq n$, $g_2 \leq n-1$, $g_3 \leq n-2$, ..., $g_k \leq n-k+1$, ...

Theorem $\#\{\text{parking functions } (f_1, \ldots, f_n)\} = (n+1)^{n-1}$.

Theorem.
$$\sum_{\substack{(f_1, \ldots, f_n) \\ \text{parking function}}} (-1)^{\binom{n+1}{2} - (f_1 + \ldots + f_n)} = A_n$$

Many classical combinatorial objects & identities can be "lifted" to the level of representations of S_n & symmetric functions.

As an example of this, we'll explain how to do this for the above theorem.

Let us define a representation of the symmetric group S_n on parking functions:

V_{PF} = the space of formal linear combinations (over $\mathbb{C}$) of parking functions $(f_1, \dots, f_n)$.

Equiv, V_{PF} = the space with basis δ_f labelled by parking functions $f = (f_1, \dots, f_n)$.

$$\dim V_{PF} = (n+1)^{n-1}.$$ ← # parking functions

some signs

$$S_n \ni w : \delta_f \mapsto \pm \delta_{w(f)},$$

where $w(f) := (f_{w^{-1}(1)}, \dots, f_{w^{-1}(n)})$

Consider (partially commutative, partially anti-commutative) variables

$$x_i^{(k)} \qquad i = 1, \dots, n; \quad k \geq 0$$

$$x_i^{(k)} \cdot x_j^{(\ell)} = (-1)^{k \cdot \ell} \, x_j^{(\ell)} \cdot x_i^{(k)}, \quad i \neq j.$$

We associate the basis elements δ_f with the monomials

$$\delta_f = x_1^{(f_1 - 1)} \cdot \ldots \cdot x_n^{(f_n - 1)}$$

S_n acts on such monomials by

$$\underset{S_n}{w \in} : x_1^{(a_1)} \cdots x_n^{(a_n)} \mapsto x_{w(1)}^{(a_1)} \cdot \ldots \cdot x_{w(n)}^{(a_n)}$$

The representation V_{PF} is graded by degrees of monomials (assuming $\deg x_i^{(k)} = k$):

$$V_{PF} = V_{PF}^{0} \oplus V_{PF}^{1} \oplus \ldots \oplus V_{PF}^{\binom{n}{2}}$$

$$V_{PF}^{k} := \text{Span of } x_1^{(f_1-1)} \cdot \ldots \cdot x_n^{(f_n-1)} \quad \text{s.t. } \sum_{i=1}^{n} (f_i - 1) = k$$

We obtain the reps. of S_n

V_{PF}^k, $k = 0, \dots, \binom{n}{2}$

Let's take the "alternating sum" of these representations.

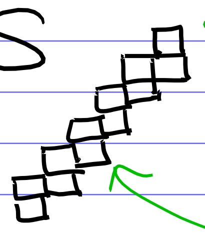
More precisely

Let $ch(V_{PF}^k)$ be the Frobenius character of V_{PF}^k.
(This is a certain symmetric function).

Now we can take an alternating sum of symm. funct.

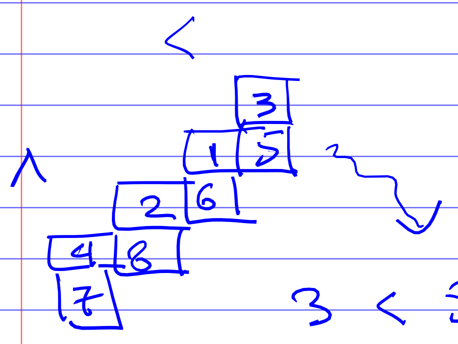
<u>Theorem</u> (Pak–P.)

$$\sum_{k=0}^{\binom{n}{2}} (-1)^{\binom{n}{2}-k} \, ch(V_{PF}^k) =$$

$= S$ 

the skew Schur function whose shape is the "zigzag ribbon" with n boxes (with 1 box in the top row)

Notice that # SYT's whose shape is this staircase ribbon is exactly the Euler number A_n (# alternating perm.)



$3 < 5 > 1 < 6 > 2 < 8 > 4 < 7$

So, taking the pricipal specialization of this theorem, we recover the identity:

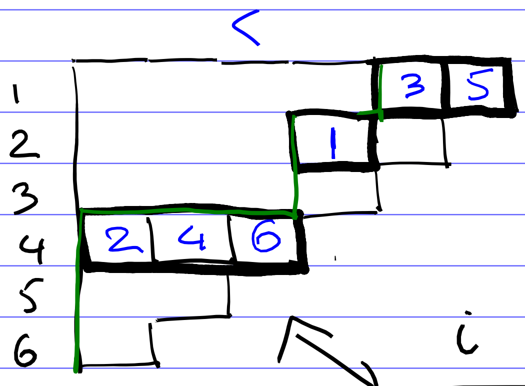
the altenating sum over parking functions $(f_1, \dots, f_n)$ $= A_n$

Let us describe certain tableaux associated with parking functions:

Lemma Parking functions of size n are in bijection with standard Young tableaux of a skew shape λ/μ s.t.

- λ/μ is a horizontal n-stip
- λ/μ fits inside the staircase shape $(n, n-1, n-2, \dots, 1)$

Example. $n = 6$



i	1	2	3	4	5	6
$f(i)$	2	4	1	4	1	4

The parking function f that corresponds to a tableau T is given by

$$f(i) = j \quad \text{if}$$

the entry i is located in j^{th} row of T.

Observation

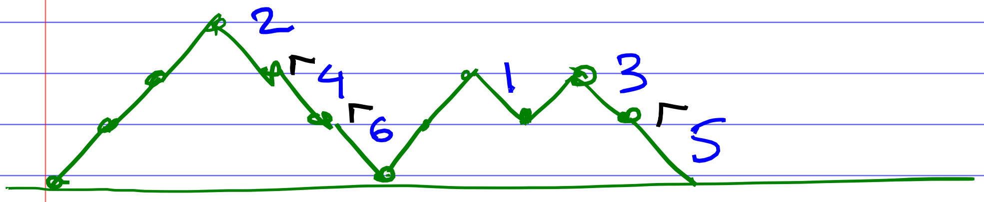
\# shapes λ/μ s.t.

- λ/μ is a horizontal n-strip
- λ/μ fits inside the staircase $(n, n-1, \dots, 1)$

equals the Catalan number C_n

Such shapes correspond to Dyck paths and parking functions can be viewed as labelled Dyck paths.

The above Example



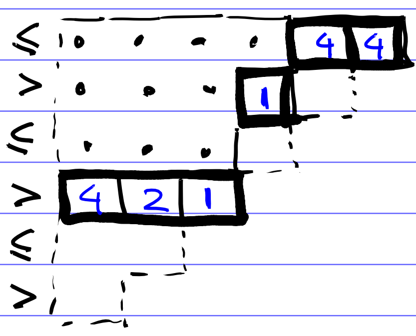
n down steps are labelled by $1, 2, \dots, n$ (w/o repetitions) s.t. the labels in any chain of consecutive down steps increase.

Let us now slightly modify these tableax:

Def. A *semi-standard parking function tableaux* is a filling of boxes on a horizontal n-strip $\lambda \subset (n, n-1, n-2, \dots, 1)$ by positive integers (with allowed repetitions) s.t.

- the entries *weakly increase* in rows $1, 3, 5, \dots$
- the entries *strictly decrease* in rows $2, 4, 6, \dots$

Example A PF-tableau



$n = 6$

$k = 10$

Then

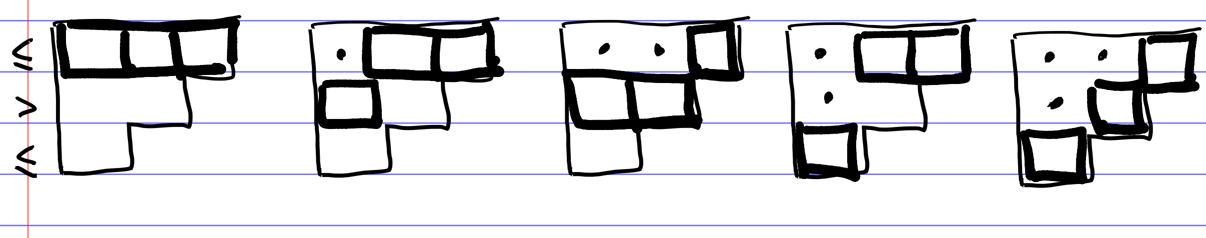
$$pf_{n,k} := ch(V_{PF}^{k})$$

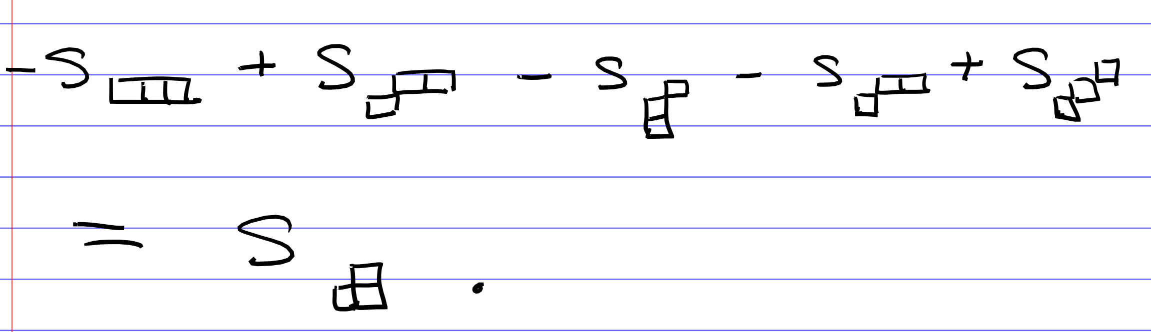
$$= \sum_{\substack{T \text{ is a semi-standard} \\ \text{parking function tableau} \\ \text{with } k \text{ boxes above it}}} x^T$$

Theorem $\sum_{k=0}^{\binom{n}{2}} (-1)^{\binom{n}{2}-k} pf_{n,k} =$

$= S_{\text{"zigzag ribbon"}}$

Example $n=3$





Generalized parking functions

$\gamma = (\gamma_1 \geq \ldots \geq \gamma_n)$

Def $(f_1, \ldots, f_n) \in \{1, 2, \ldots\}^n$ is a **γ-parking function** if its decreasing rearrangement $g_1 \geq g_2 \geq \ldots \geq g_n$ satisfies

$$g_i \leq \gamma_i \quad \forall\, i = 1, \ldots, n.$$

Exercise If γ is linear, i.e. if $\gamma = ((n-1)a+b, (n-2)a+b, \ldots, 2a+b, a+b, b)$

then # γ-parking functions equals $(an+b)^{n-1}$.

All above constructions (parking function reps. of S_n, parking function tableaux & symm. functions) can be extended to γ-parking functions ...

Def A <u>semi-standard</u> <u>γ-PF-tableau</u> is a filling of the shape λ/μ s.t.

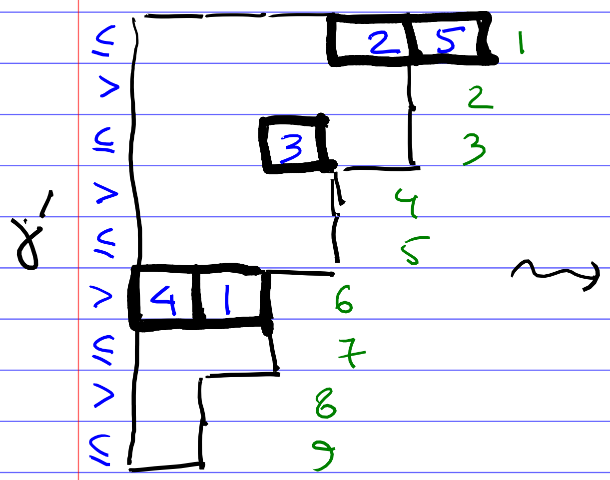
- λ/μ is a horizontal n-strip
- λ/μ fits inside γ' (the conj. partition)

satisfying the same conditions as before

We define reps of S_n & corresp. symmetric functions as before:

$$pf_{\gamma,k} := \sum_{\substack{T\ \gamma\text{-PF-tableau} \\ \text{with } k \text{ empty} \\ \text{boxes above it}}} x^T \subset \Lambda$$

Example $\gamma = (9, 7, 5, 3, 1)$
$n = 5$



i	1	2	3	4	5
f_i	6	1	3	6	1

γ-parking function

a <u>standard</u> γ-PF-tableau

Theoreom (P.-P.)

$$(-1)^{|\gamma|-n} \cdot \sum_{k\geq 0} (-1)^k \, pf_{\gamma,k}$$

$$= \begin{cases} 0 & \text{if } \gamma_n \text{ is even} \\ S_{\varkappa} & \text{if } \gamma_n \text{ is odd} \end{cases}$$

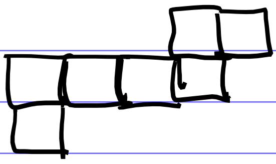
where $\varkappa$ is the ribbon shape with n boxes constructed, as follows:

- start with a box
- for $i = 2, 3, \ldots, n$ add the i^{th} box below, resp. to the left, of the $(i-1)^{st}$ box, if γ_{n+1-i} is even, resp. odd.

Example $\quad \gamma = (10, 9, 9, 7, 6, 5, 3)$

$$\begin{array}{ccccccc} e & o & o & o & e & o & o \\ \downarrow & \leftarrow & \leftarrow & \leftarrow & \downarrow & \leftarrow & \end{array}$$

$\varkappa =$



Proof (Involution Principle)

We'll construct a sign-reversing involution on semi-standard γ – PF-tableaux T

Let σ_i be the entry of T in column i,
Assign the direction "up" or "down" for each entry σ_i:

$$\varepsilon_i := \operatorname{sgn}(\sigma_i - \sigma_{i+1})(-1)^{\operatorname{row}(\sigma_i)}$$

"down" if $+1$
"up" if -1

$\begin{cases} 1 & \text{if } \sigma_i > \sigma_{i+1} \\ -1 & \text{if } \sigma_i < \sigma_{i+1} \end{cases}$

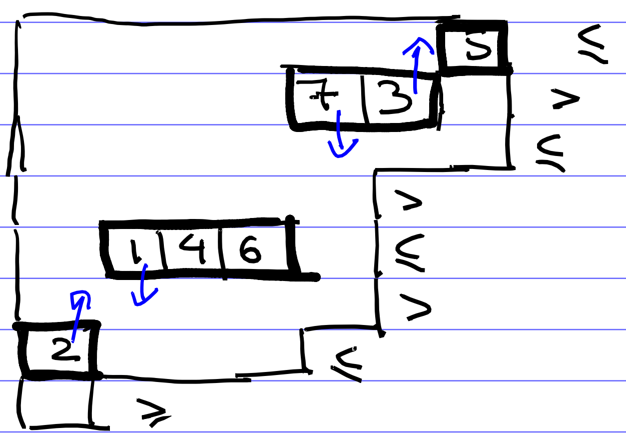
(assuming $\sigma_{n+1} = n+1$)

the index of row containing σ_i

We say that a box of T is movable down if moving it in the assigned direction would produce a valid γ- P.f -tableau.

Example $n=7$, $\gamma = (8,7,7,7,6,3,3)$

γ'



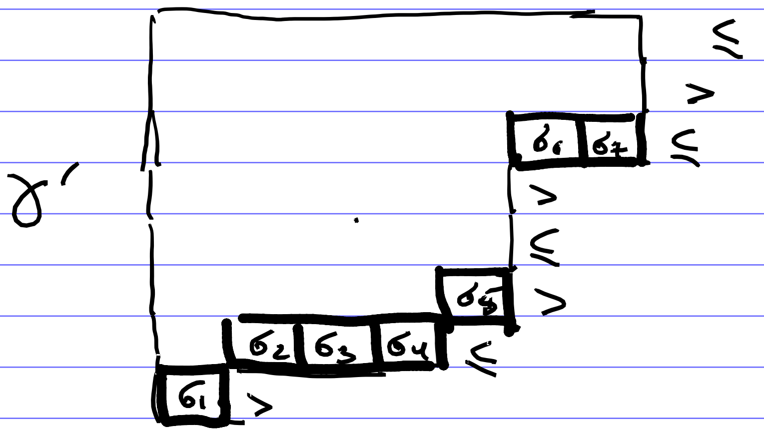
Involution: If T has at least one movable box, then move the rightmost movable box.

Easy to see:

- this map changes the parity of k = # boxes above T.
- this is an involution on the set of all semi-std. γ-p.f.-tableaux with at least 1 movable box.

Tableaux w/o movable boxes are of the following form:

Ex.



where

$$\sigma_1 > \sigma_2 \leq \sigma_3 \leq \sigma_4 \leq \sigma_5 > \sigma_6 \leq \sigma_7$$

(there are no such tableaux if γ_n is even $\Rightarrow$ box σ_n is always movable).

Such $\sigma_1 \dots \sigma_n$ are exactly the entries of a SSYT of ribbon shape $\varkappa$. $\square$