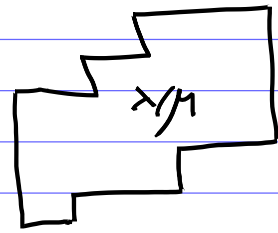
Hook length formula for skew Young diagrams λ/μ?



In general,

#SYT's of skew shape λ/μ is not given by a simple product formula.

This number might involve large prime factors when parts of λ & μ are relatively small.

So for a long time people thought that there is no hook length formula for skew shapes...

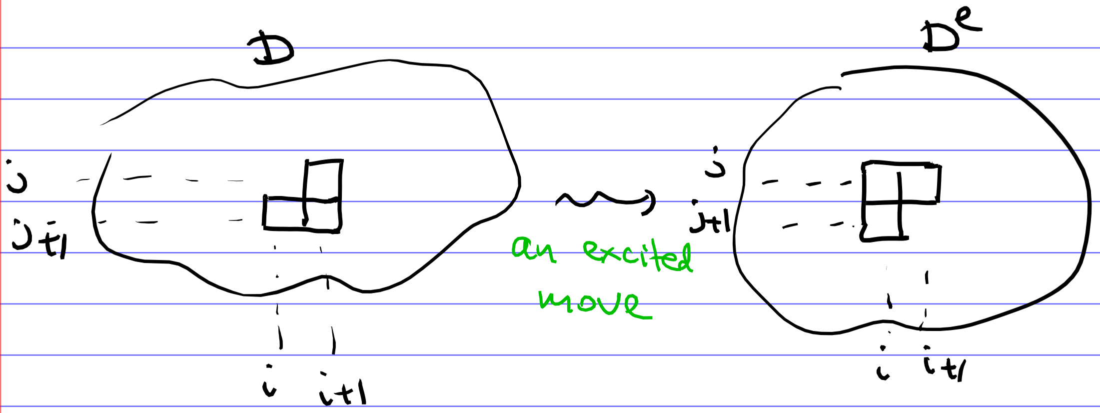
But in 2014, Naruse found a generalization of hook length formula to skew shapes.

Let D be any subset of boxes on the plane (not necessarily a skew shape).

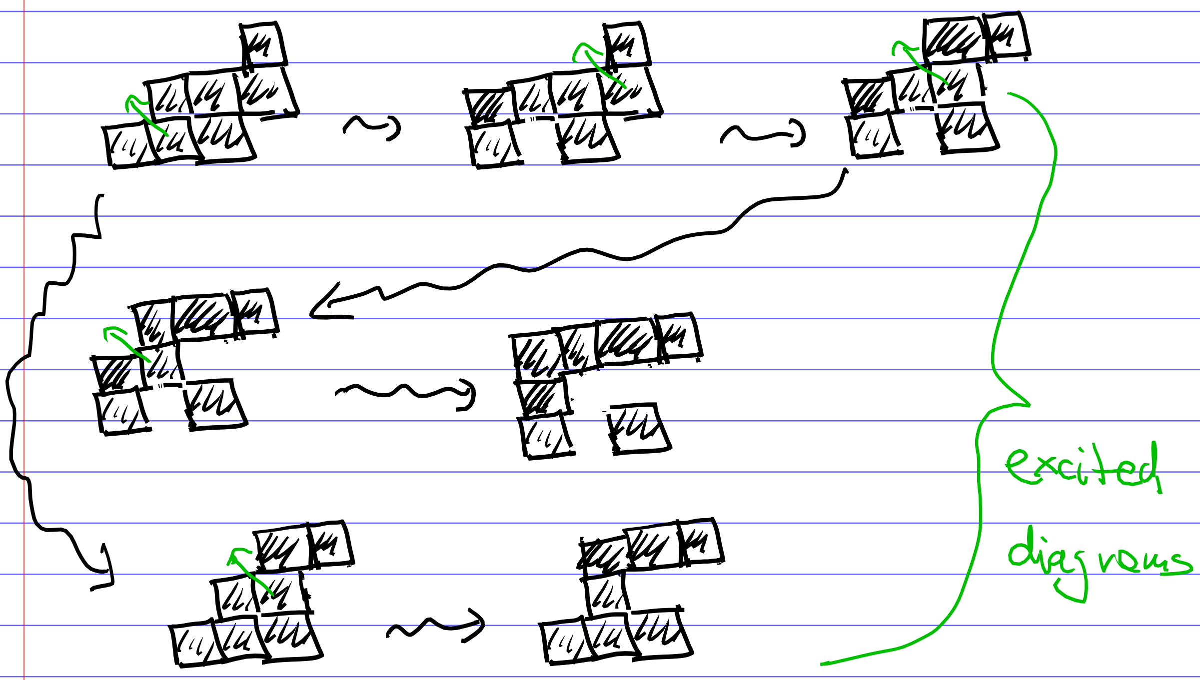
Def. Excited moves

$$D \rightsquigarrow D^e$$

if $(i+1, j+1), (i+1, j), (i, j+1) \in D$ and $\tilde{D}$ is obtained from D^e by replacing the box $(i+1, j+1)$ with the box (i, j)



Examples



All diagrams obtained from λ/μ by a sequence (possibly empty) of excited moves are called excited diagrams for λ/μ

Naruse hook length formula

Let λ/μ be a skew shape with $|\lambda/\mu| = n$ boxes.

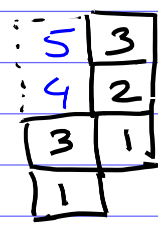
$$f_{\lambda/\mu} = n! \left(\sum_{\substack{D \text{ excited} \\ \text{diagram} \\ \text{for } \lambda/\mu}} \prod_{a \in D} \frac{1}{h(a)} \right)$$

\# SYT's of shape λ/μ

the usual hook length of the box a in shape λ.

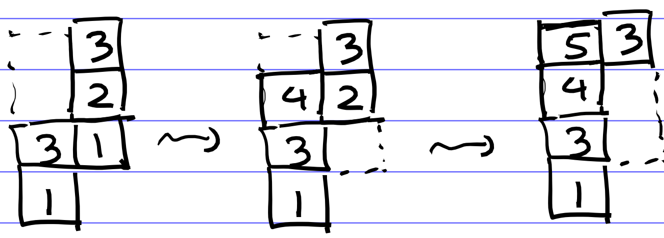
Example.

$$\lambda/\mu = (2,2,2,1)/(1,1) =$$



hook lengths of all boxes in λ

Excited diagrams:



$$f_{(2221)/(1,1)} =$$

$$= 5! \left(\frac{1}{3\cdot 3\cdot 2\cdot 1\cdot 1} + \frac{1}{4\cdot 3\cdot 2\cdot 2\cdot 1} + \frac{1}{5\cdot 4\cdot 3\cdot 3\cdot 1} \right)$$

$$= 9.$$

There are several things that I usually mention in any combinatorics class that I teach (because they appear in any area of combinatorics):

- the Catalan numbers
- parking functions

We already mentioned the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ in this class.

(For example, C_n = # SYTs of the rectangular shape $2 \times n$.

So the hook length formula is a generalization of the formula for the Catalan numbers.)

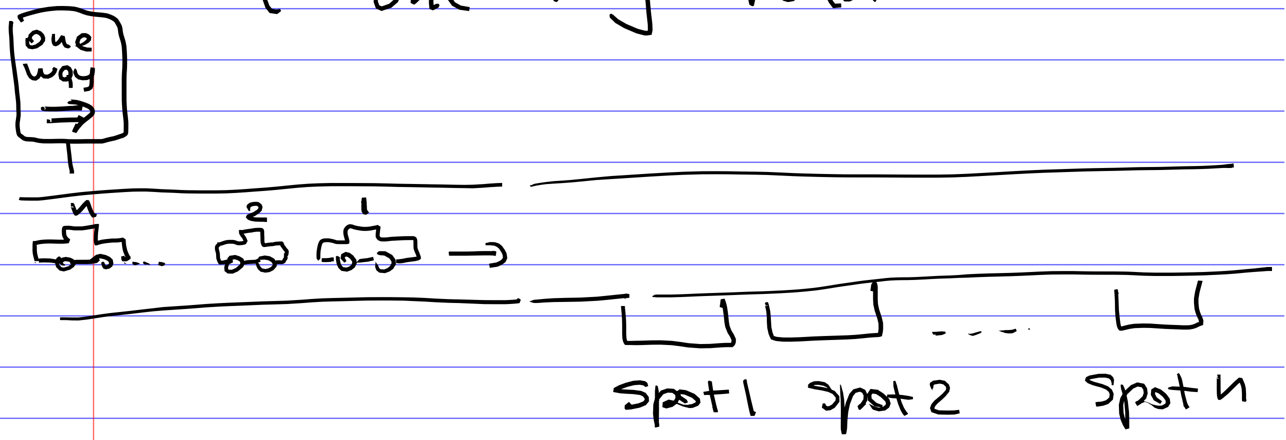
Let's talk about parking functions ...

Parking functions

It can be hard to find a parking spot for your car. Parking functions correspond to situations when we can park n cars.

We have

- n cars
- n parking spots on a one way road



Preference function

$$f : [n] \to [n]$$

The driver of i^{th} car prefers to park in the $f(i)^{th}$ spot.

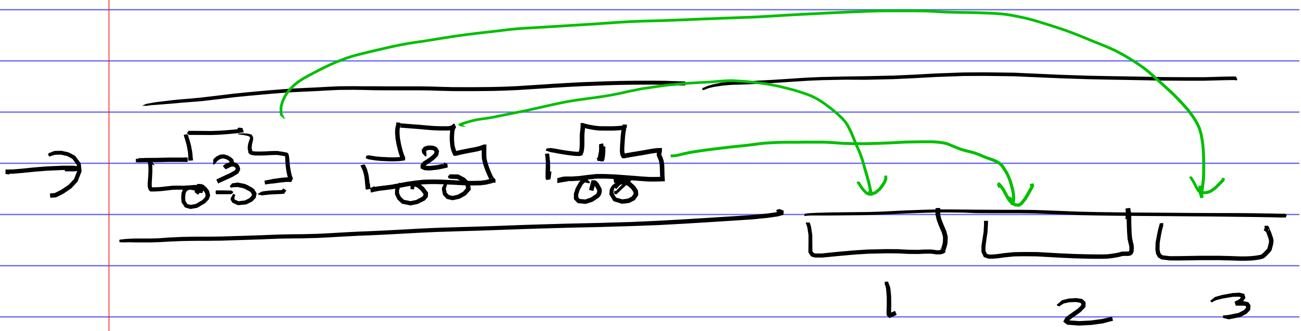
The cars perk, as follows:

- 1^{st} car perk in the spot $f(1)$.
- 2^{nd} car drives to the spot $f(2)$. If it is empty, it perks there. But if it is alredy occupied, then it keeps driving untill it find the first available spot
- 3^{rd}, 4^{th}, ... cars do the same (drive to their prefered spots, try to perk there (if they can), otherwise keep driving untill they find an available spot)

<u>Def</u> A function $f: [n] \to [n]$ is called a <u>perking function</u> if all n cars can park using this procedure.

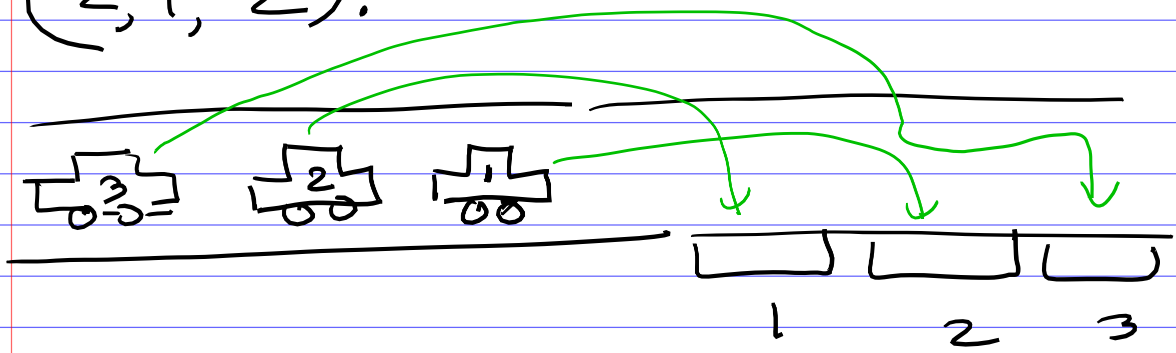
Example $n = 3$

$(f(1), f(2), f(3)) = (2, 1, 3)$



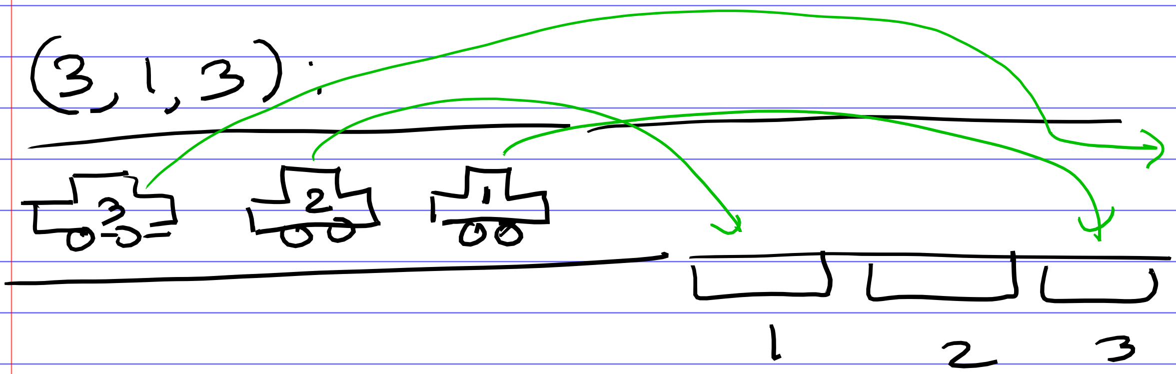
everybody parked so $(2,1,3)$ is a parking function. Actually any permutation of $1,2,..,n$ is a parking function.

$(2,1,2)$:



$(2,1,2)$ is also a parking funct.

$(3,1,3)$:



3^{rd} car is out of luck

So $(3,1,3)$ is <u>not</u> a parking function.

Lemma $f(1), \dots, f(n) \in [n]$

TFAE:

- $f(1), \dots, f(n)$ is a parking function.

- the sequence contains at most one entry n, at most 2 entries $\geq n-1$, at most 3 entries $\geq n-2$, ...
  at most k entries $\geq n-k+1$ for $k = 1, 2, \dots, n$.

- There exists a permutation w on $1, 2, \dots, n$ such that $f(i) \leq w(i)$ for $i = 1, \dots, n$.

Exercise: Prove this Lemma.

Example Parking functions for $n = 3$.

all permutations:

$(1,2,3)$, $(2,1,3)$, $(1,3,2)$, $(3,1,2)$
$(2,3,1)$, $(3,2,1)$

and everything "below" permutations

$(1,2,2)$, $(2,1,2)$, $(2,2,1)$
$(1,1,3)$, $(1,3,1)$, $(3,1,1)$
$(1,1,2)$, $(1,2,1)$, $(2,1,1)$
$(1,1,1)$

In total, we have

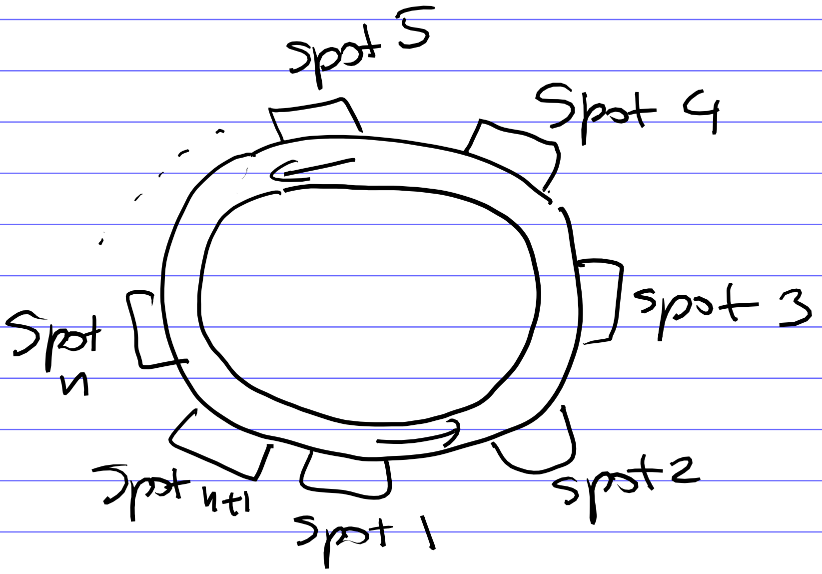
$6 + 3 + 3 + 3 + 1 = 16$

parking functions for $n = 3$.

<u>Theorem</u> There are exactly $(n+1)^{n-1}$ parking functions of size n.

<u>Proof</u>. Let us slightly modify the setup.

- n cars
- $n+1$ parking spots on a <u>circular</u> road
- preference function $f: [n] \to [n+1]$



Now any function $f:[n] \to [n+1]$ produces a valid parking of the cars. Moreover, one parking spot will be left empty.

This construction has a circular symmetry. Basically, all parking spots are the same. (There are no first and no last parking spots).

Let F_i (for $i \in [n+1]$) be the set of functions $f : [n] \to [n+1]$ that produce a parking of the cars where i^{th} spot is left empty.

Because of the circular symmetry,

$$|F_1| = |F_2| = \ldots = |F_{n+1}|.$$

Now, the total # of all functions $f : [n] \to [n+1]$ equals $(n+1)^n$.

So $|F_i| = \dfrac{(n+1)^n}{n+1} = (n+1)^{n-1}$ for any i.

Now observe that F_{n+1} is exactly the set of parking functions of size n.

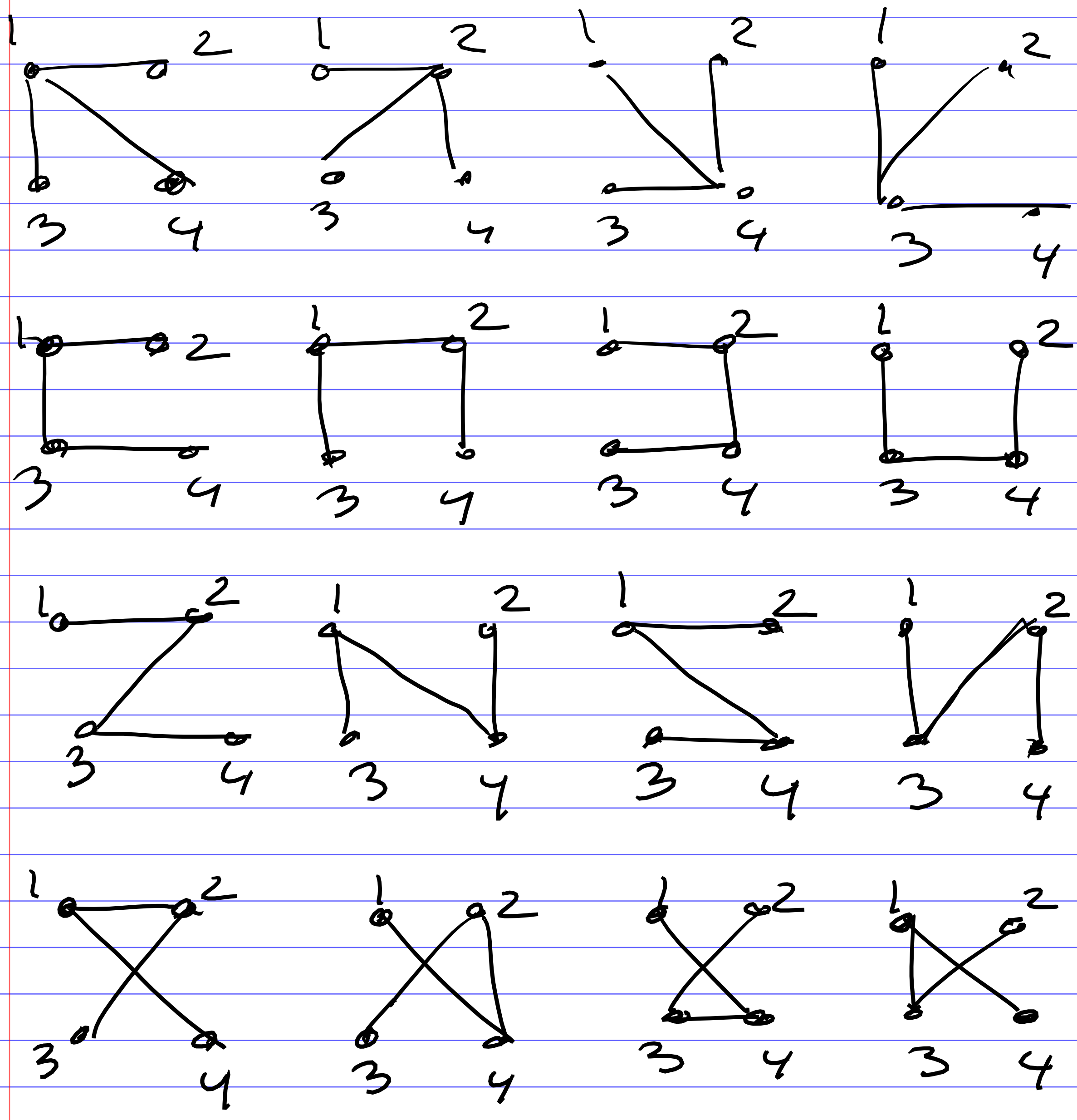
no car drives past the spot $i = n+1$ & all n cars are parked in the spots $1, \ldots, n$

$f(1), \ldots, f(n) \Longleftrightarrow$ is a parking function.

$\square$

$(n+1)^{n-1}$ is the <u>Cayley formula</u> for the number of trees on $n+1$ labelled vertices.

<u>Ex.</u> $n=3$



16 labelled trees on 4 vertices.

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parking functions $\xleftrightarrow{\text{bijection?}}$ labelled trees

<u>Exercise.</u> Construct such a bijection.

There is a natural statistics on parking functions

$$S(f) := f(1) + \dots + f(n).$$

Actually it is a little bit more convenient to consider the statistics

$$f \longmapsto \binom{n+1}{2} - S(f).$$

$= 0$ if f is a permutation

$= 1$ if f is a permutation with one entry decreased by 1

$\dots$

$= \binom{n}{2}$ if f is the smallest parking function $(1, 1, \dots, 1)$.

Tree inversion polynomial

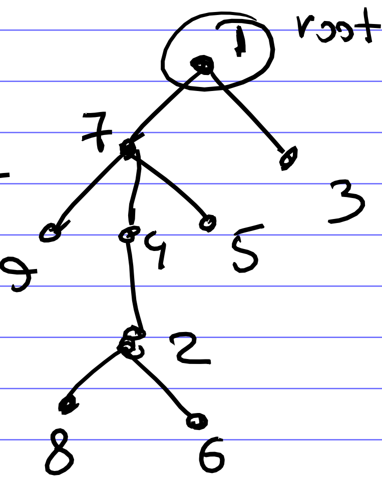
Let T be a tree on vertices $1, 2, \ldots, n+1$.
Assume that the vertex 1 is the root.

A pair (i,j) $i,j \in \{2, \ldots, n+1\}$ is an <u>inversion</u> of T if $i<j$ and the vertex j belongs to the shortest path from the vertex i to the root $\underline{1}$.

Let $\mathrm{inv}(T) := \#$ inversions in T

Example

$T =$



inversions:

$(4,7)$, $(5,7)$

$(2,7)$, $(2,4)$

$(6,7)$

$\mathrm{inv}(T) = 5.$

Tree inversion polynomial is

$$I_n(x) = \sum_{\substack{T \text{ labelled} \\ \text{tree on } n+1 \\ \text{vertices } 1,2,\ldots,n+1}} x^{\mathrm{inv}(T)}.$$

If you know what the Tutte polynomial $T_G(x,y)$ is...

$$I_n(y) = T_{K_{n+1}}(1,y)$$

Theorem (Kreweras)

$$I_n(x) = \sum_{\substack{f \text{ parking} \\ \text{function of size } n}} x^{\binom{n+1}{2} - (f(1)+\cdots+f(n))}.$$

An interesting identity.

Def. An alternating permutation $w \in S_n$ is a permutation s.t. $w_1 < w_2 > w_3 < w_4 > \cdots$

$A_n :=$ # of alternating permutations.

The numbers A_n are known under many different names (Euler numbers, André numbers, zigzag number, up/down numbers, tangent and secant numbers, ...)

They are related to the Bernoulli numbers.

__Theorem__ $I_n(-1) = A_n$.

In other words,

$$\sum_{\substack{f \text{ parking function} \\ \text{of size } n}} (-1)^{\binom{n+1}{2} - (f(1)+\dots+f(n))}$$

$$= A_n.$$

__Example__ $n=3$

Parking functions:

6 permutations

$3+3$ functions obtained by perm of $(1,2,2)$ or $(1,1,3)$

3 functions obtained by perm. of $(1,1,2)$

1 minimal park funct. $(1,1,1)$

We get $6-(3+3)+3-1=2$

$= A_3 = \#$ alternating permutations:

$1<3>2$, $2<3>1$.

One way to prove this is to show that both sides satisfy the same recurrence relation...

- Can we understand this understand this identity on a more conceptual level?

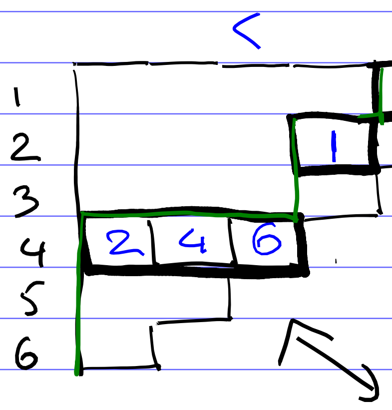
- How is all this stuff (parking functions, alternating permutations) related to the topics of this course (symmetric functions, representations of S_n, Young tableaux, ...)

Let us give another way to graphically represent a parking function.

<u>Theorem</u> Parking functions of size n are in bijection with standard Young tableaux of a skew shape λ/μ s.t.

- λ/μ is a horizontal n-stip
- λ/μ fits inside the staircase shape $(n, n-1, n-2, \dots, 1)$

Example. $n = 6$



i	1	2	3	4	5	6
$f(i)$	2	4	1	4	1	4

The parking function f that corresponds to a tableau T is given by

$f(i) = j$ if

the entry i is located in j^{th} row of T.

<u>Observation</u>

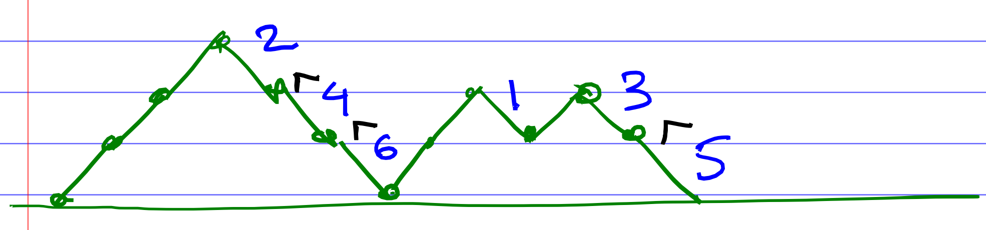
\# shapes λ/μ s.t.

- λ/μ is a horizontal n-strip
- λ/μ fits inside the staircase $(n, n-1, \ldots, 1)$

equals the <u>Catalan number C_n</u>

Such shapes correspond to <u>Dyck paths</u> and parking functions can be viewed as labelled Dyck paths.

<u>The above Example</u>

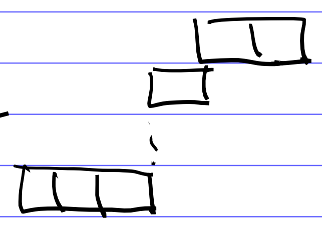
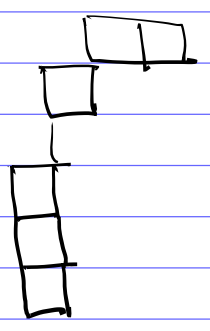


n <u>down</u> steps are labelled by $1, 2, \ldots, n$ (w/o repetitions) s.t. the labels in any chain of consecutive down steps <u>increase</u>.

Let's transform these objects into symmetric functions...

For a horizontal n-strip λ/μ that fits inside $(n, n-1, \ldots, 1)$, let $\widetilde{\lambda/\mu}$ be the skew shape obtained from λ/μ by leaving rows $1, 3, 5, \ldots$ as they are and transforming rows $2, 4, 6, \ldots$ into columns

Ex.

$\lambda/\mu =$  $\rightsquigarrow \widetilde{\lambda/\mu} =$ 

Consider the "parking functions" symmetric function in Λ:

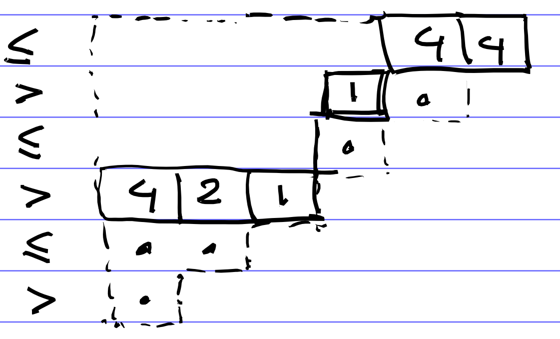
$$pf_{n,k} := \sum_{\substack{\lambda/\mu \text{ horizontal} \\ n\text{-strip} \subset (n, n-1, \ldots, 1) \\ \binom{n}{2} - |\mu| = k}} s_{\widetilde{\lambda/\mu}}$$

Equivalently, define a semi-standard PF-tableau as a filling of a horizontal n-strip $\subset (n, n-1, \ldots, 1)$ by positive integers (repetitions are allowed) s.t.

- the entries weakly increase in rows $1, 3, 5, \ldots$
- the entries strictly decrease in rows $2, 4, 6, \ldots$

stands for "parking functions"

Example A PF-tableau



$n = 6$

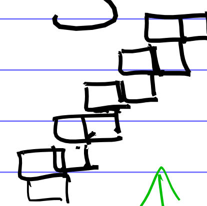
$K = 5$

Then

$$p_{n,k} = \sum_{\substack{T \text{ is a semi-standard} \\ \text{PF-tableau that} \\ \text{fits inside } (n, n-1, \ldots, 1) \\ \text{s.t. the area of} \\ \text{the staircase shape } (n, \ldots, 1) \\ \text{below } T \text{ consists} \\ \text{of } K \text{ boxes}}} x^T$$

$p_{n,k}$ can also be defined as the Frobenius character of a certain representation of S_n on parking functions f of size n with $K = \binom{n+1}{2} - (f(1) + \ldots + f(n))$.

Theorem (Pak–P.)

$\sum_{k=0}^{\binom{n}{2}} (-1)^k \, pf_{n,k}$ equals S 

the skew Schur function for the zigzag ribbon with n boxes

all rows, except the last, have 2 boxes. The last row has either 1 or 2 boxes (depending on parity of n

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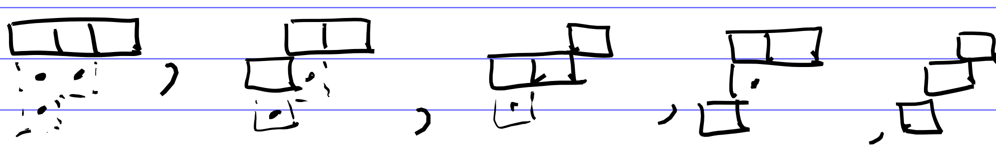
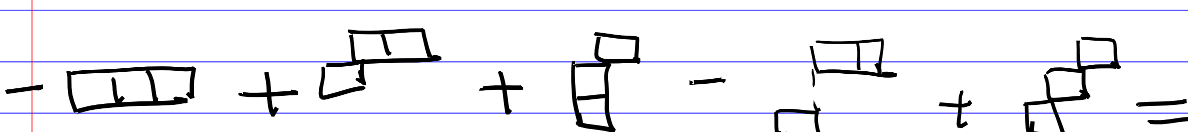
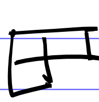
Clearly, the standard Young tableaux of this "zigzag ribbon" shape are in bijection with alternating permutations.

Example $n=3$





$=$ 

Generalized parking functions.

We can replace the staircase $(n, n-1, \ldots, 1)$ by any Young diagram $(\nu_1, \nu_2, \ldots, \nu_\ell)$ with $\nu_1 = n$

Let $\gamma = (\gamma_1, \ldots, \gamma_n) = \nu'$ (the conjugate partition)

Def A γ-parking function is a function

$$f : [n] \to \{1, 2, 3, \ldots\} \text{ s.t.}$$

the weakly decreasing rearrangement $\beta_1 \geq \ldots \geq \beta_n$ of $f(1), \ldots, f(n)$ satisfies $\beta_i \leq \gamma_i \quad \forall i.$

Exercise, For $n, a, b \geq 1$ Prove that #

γ-parking functions for

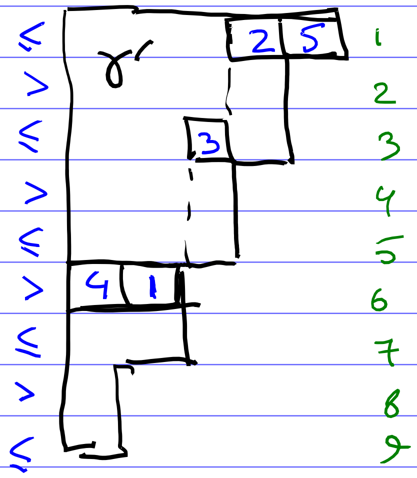
$$\gamma = ((n-1)a+b, (n-2)a+b, \ldots, 2a+b, a+b, b)$$

equals $(an+b)^{n-1}$.

γ-parking functions are in bijection with standard tableau of shape λ/μ s.t.

- λ/μ is a horizontal n-strip
- $\lambda/\mu \subset \nu = \gamma'$

Example $\gamma = (9, 7, 5, 3, 1)$



$\rightsquigarrow$

i	1	2	3	4	5
$f(i)$	6	1	3	6	1

The total number of p.f's for this γ is

$$(2\cdot 5+1)^4$$

How about the alternating sum of #'s of γ-parking functions?

Define the symmetric function $pf_{\gamma,k}$ in the same way as $pf_{n,k}$ but replace the staircase by Young diagram $\nu = \gamma'$.

Theorem (P.P.)

$$\sum_{k\geq 0} (-1)^k pf_{\gamma,k} \text{ equals}$$

- 0 if γ_n is even
- $s_{\varkappa}$ if γ_n is odd

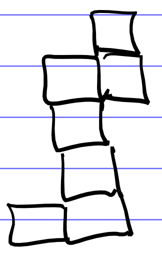
and $\varkappa$ is the ribbon with n boxes constructed as follows:

- start with a box
- $i = 2, 3, \ldots, n$ add the ith box to the left/below of the previous box if γ_{n+1-i} is even/odd.

Example

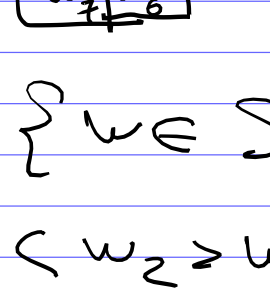
$$\gamma = (10\ 9\ 9\ 7\ 6\ 5\ 3)$$

$$\begin{array}{ccccccc} e & o & o & o & e & o & o \\ \leftarrow & \downarrow & \downarrow & \downarrow & \leftarrow & \downarrow & = \end{array}$$

$\varkappa =$ 

In particular, the alternating sum of the numbers of γ-parking functions (with the sign $(-1)^{|\gamma| - (f(1) + \ldots + f(n))}$) equals # SYT of shape $\varkappa$, i.e. # permutations $w \in S$ with prescribed positions of descents & ascents.

For the above example we get



$$\#\{w \in S_7 \mid w_1 < w_2 > w_3 < w_4 < w_5 < w_6 > w_7\}$$