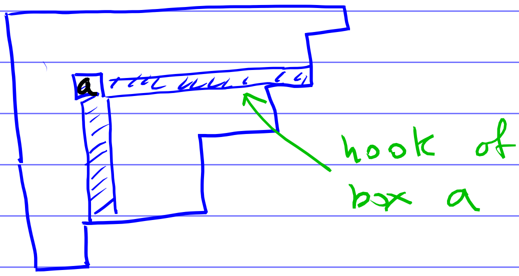
... More on the hook length formula...

$f_\lambda :=$ # SYT's of shape λ.

Hook length formula: $\lambda \vdash n$

$$f_\lambda = n! \Big/ \prod_{a \in \lambda} h(a)$$

hook length of box a in λ



We've proved it using either

- Hillman-Grassl correspondence
- (generalized) RSK correspondence.

Other ways to prove the hook length formula:

- Gessel-Viennot method $\Rightarrow$
  $\Rightarrow$ determinantal formula for f_λ $\Rightarrow$
  $\Rightarrow$ hook length formula.
- Weyl's dimension formula $\Rightarrow$
  $\Rightarrow$ hook-content formula $\Rightarrow$
  $\Rightarrow$ hook length formula.

<u>Today</u>: A <u>probabilistic</u> <u>proof</u> of the hook length formula, or

Hook walk proof

due to Greene, Nijenhuis, Wilf, 1979.

- This proof uses hooks & explains why the hook lengths appear in the formula.
- Zeilberger converted this proof into a bijective proof.

Let $H(\lambda) := \prod_{a \in \lambda} h(a)$

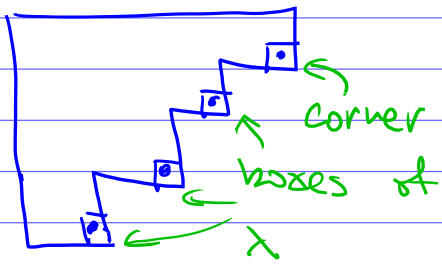
<u>Hook length formula</u>: $f_\lambda = \dfrac{n!}{H(\lambda)}$.

Clearly, f_λ is defined by the recurrence relation:

$$f_\lambda = \sum_{\mu : \mu \lessdot \lambda} f_\mu$$

μ is obtained from λ by removing a single corner box

$$f_\emptyset = 1$$



For an SYT T of shape $\lambda \vdash n$, let $\tilde{T}$ be the SYT obtained from T by removing the box filled with n. Then $\tilde{T}$ is an SYT of shape $\mu \lessdot \lambda$.

$$T = \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & 7 \\ 6 & & \end{array} \rightsquigarrow \tilde{T} = \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & \\ 6 & & \end{array}$$

In order to prove the hook length formula, it is enough to show that the same recurrence relation holds for the right hand side.

Need to show:

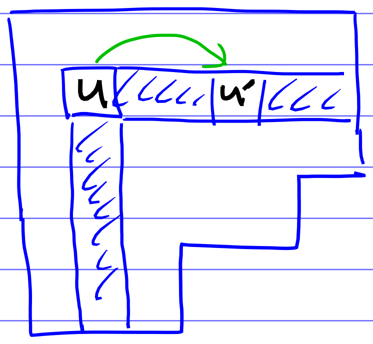
$$\boxed{\frac{n!}{H(\lambda)} = \sum_{\mu \lessdot \lambda} \frac{(n-1)!}{H(\mu)}}$$

Clearly, $\frac{0!}{H(\emptyset)} = 1$ ← base of recurrence

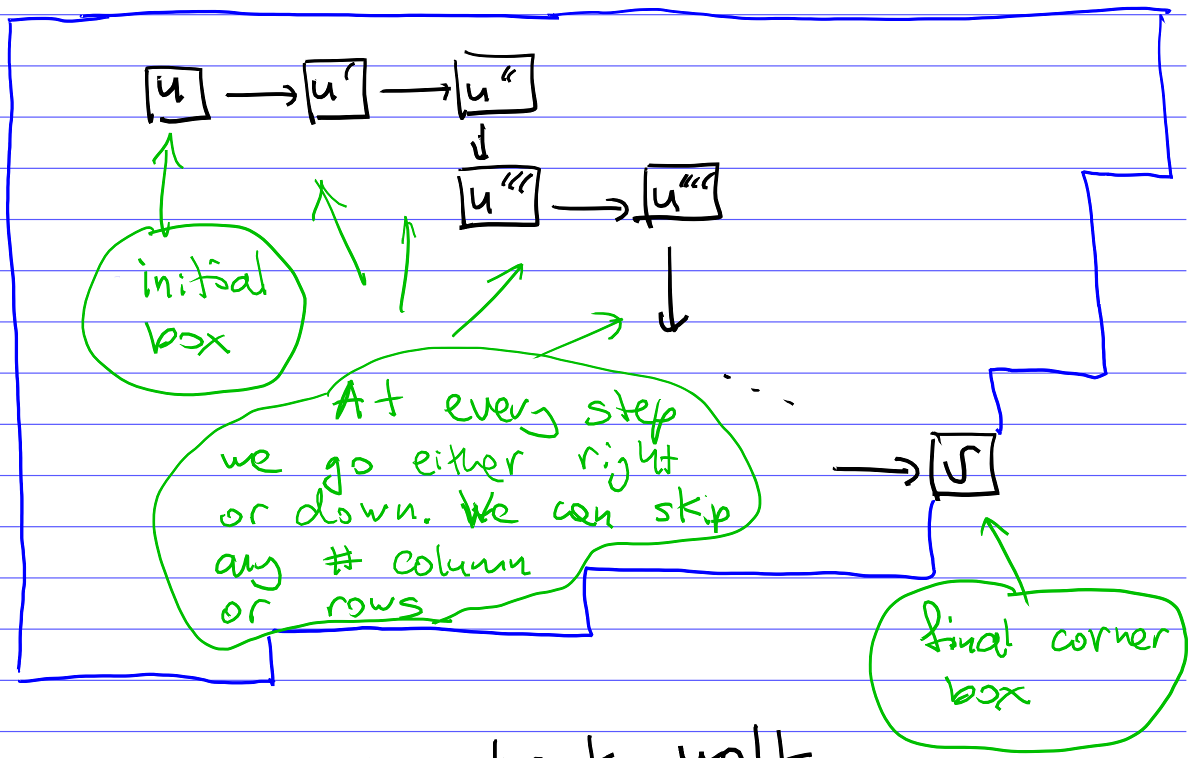
Fix a Young diagram $\lambda \vdash n$.

The hook walk is a random walk on boxes of the Young diagram λ s.t.

- Pick an initial box $u \in \lambda$ with uniform probability $= \frac{1}{n}$.
- If u is not a corner box, then jump from box u to any other box $u' \neq u$ in the hook of u with uniform probability $\frac{1}{h(u)-1}$.



- Then jump from u' to any other box u'' in the hook of u', etc.
- Repeat until we arrive to a corner box δ.
- Stop.

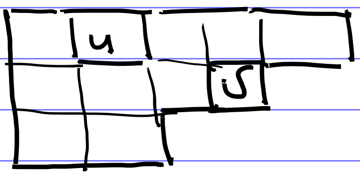


a hook walk

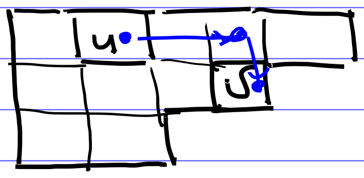
$P(u,\sigma) :=$ the probability that a hook walk starting at box u ends at corner σ.

$$P(u,\sigma) = \sum_{\substack{\text{paths} \\ u \to u' \to u'' \to \dots \to \sigma}} \frac{1}{h(u)-1} \cdot \frac{1}{h(u')-1} \cdot \frac{1}{h(u'')-1} \cdots$$

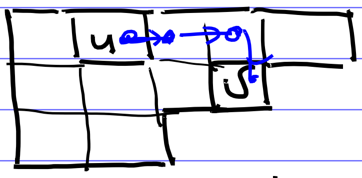
Example



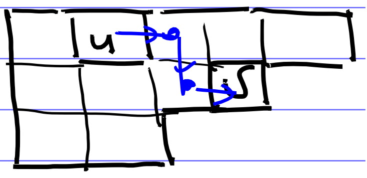
hook walks:



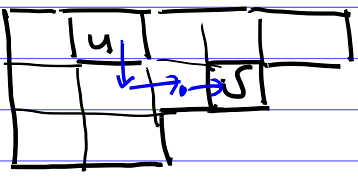
$\frac{1}{(6-1)} \cdot \frac{1}{(3-1)}$



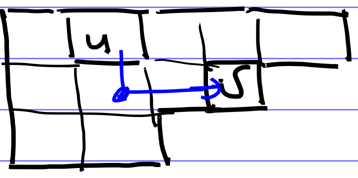
$\frac{1}{(6-1)} \cdot \frac{1}{(4-1)} \cdot \frac{1}{(3-1)}$



$\frac{1}{(6-1)} \cdot \frac{1}{(4-1)} \cdot \frac{1}{(2-1)}$



$\frac{1}{(6-1)} \cdot \frac{1}{(4-1)} \cdot \frac{1}{(2-1)}$



$\frac{1}{(6-1)} \cdot \frac{1}{(4-1)}$

$$P(u,\sigma) = \frac{1}{(6-1)} \cdot \frac{1}{(3-1)} + \frac{1}{(6-1)} \cdot \frac{1}{(4-1)} \cdot \frac{1}{(3-1)} +$$

$$+ \frac{1}{(6-1)} \frac{1}{(4-1)} \cdot \frac{1}{(2-1)} + \frac{1}{(6-1)} \cdot \frac{1}{(4-1)} \cdot \frac{1}{(2-1)} +$$

$$+ \frac{1}{(6-1)} \cdot \frac{1}{(4-1)}.$$

Let $P(s) :=$ the probability that a hook walk ends at corner box s.

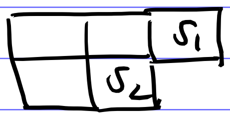
Clearly, $$P(s) = \sum_{\substack{u \text{ any} \\ \text{box of } \lambda}} \frac{1}{n} \cdot P(u,s)$$

the prob. to pick an initial box u.

We have

$$\sum_{\substack{s \text{ any } \underline{\text{corner box}} \\ \text{of } \lambda}} P(s) = 1$$

the sum of all probabilities adds to 1

Example. 

$$P(s_1) = \frac{1}{5} + \frac{1}{5} \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{1}{3} + \frac{1}{5} \frac{1}{3} \frac{1}{2}$$

$$= \frac{1}{5}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2}\right)$$

$$= \frac{1}{5}\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)$$

$$= \frac{1}{5} \frac{3}{2} \cdot \frac{4}{3} = \boxed{\frac{2}{5}}.$$

$$P(s_2) = \frac{1}{5} + \frac{1}{5} \cdot \frac{1}{1} + \frac{1}{5} \cdot \frac{1}{2}$$

$$+ \frac{1}{5} \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{5} \frac{1}{3} \cdot \frac{1}{1}$$

$$= \frac{1}{5}\left(1 + 1 + \frac{1}{2} + \frac{1}{3} \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{1}\right)$$

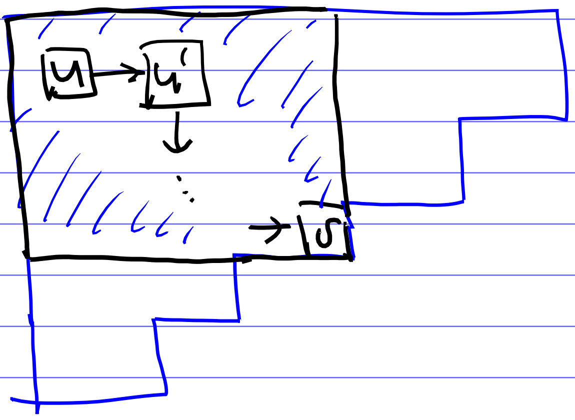
$$= \frac{1}{5}\left(1 + \left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\right) = \boxed{\frac{3}{5}}.$$

$$\frac{2}{5} + \frac{3}{5} = 1$$

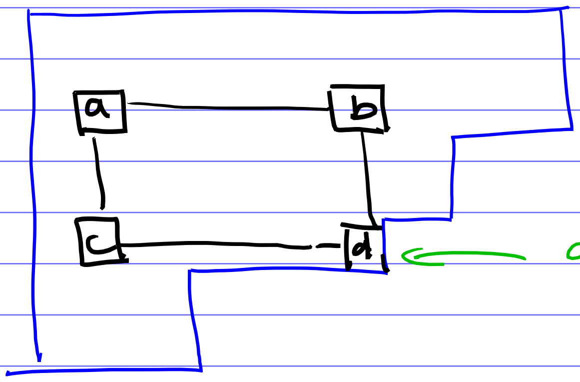
Let us calculate these probabilities in a different way...

<u>Some observations:</u>

(1) If we fix a corner box δ, then any hook walk ending at δ belongs to the rectangle.



(2)



$$h(a) + h(d) = h(b) + h(c)$$

$$(h(a)-1) + \underbrace{(h(d)-1)}_{=0} = (h(b)-1) + (h(c)-1)$$

Define the weight of a box in λ

$$wt(a) := \frac{1}{h(a)-1}$$

If $wt(b) = \frac{1}{x}$ & $wt(c) = \frac{1}{y}$,

then $wt(a) = \frac{1}{x+y}$.

Let us prove a couple of lemmas about paths in a rectangle with arbitrary weights of this form.

Consider the $(k+1)\times(\ell+1)$ rectangle with boxes weighted, as follows:

				i			
	$\frac{1}{x_1+y_1}$	$\frac{1}{x_2+y_1}$	$\cdots$	$\vdots$	$\cdots$		$1/y_1$
	$\frac{1}{x_1+y_2}$	$\frac{1}{x_2+y_2}$	$\cdots$		$\cdots$		$1/y_2$
				$\vdots$			$\vdots$
j	$\cdots$	$\cdots$	$\cdots$	$\frac{1}{x_i+y_j}$	$\cdots$	$\cdots$	$1/y_j$
				$\vdots$			$\vdots$
				$\vdots$			$1/y_\ell$
	$1/x_1$	$1/x_2$	$\cdots$	$1/x_i$	$\cdots$	$1/x_k$	1

Here $x_1, \dots, x_k, y_1, \dots, y_\ell$ are veriables.

The box in row j & column i hox weight $\frac{1}{x_i+y_j}$ for $i \in [k]$, $j \in [\ell]$

Boxes in the last row have weights $\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_k}; 1$

Boxes in the last column have weights $\frac{1}{y_1}, \frac{1}{y_2}, \dots, \frac{1}{y_\ell}, 1$

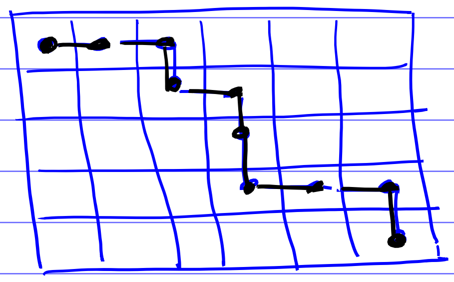
For any path P,

$\mathrm{wt}(P) :=$ the product of weights of boxes in P

Lemma 1. $\displaystyle\sum_{\substack{P \text{ is a lattice path from the} \\ \text{upper left to the lower right box.}}} \mathrm{wt}(P) = \frac{1}{x_1 \dots x_k y_1 \dots y_\ell}$

the sum is over $\binom{k+\ell}{\ell}$ lattice paths in the rectangle, not over hook walks. Here we cannot skip over rows or columns

P



$(\ell+1) \times (k+1)$ rectangle

Example. $k = \ell = 1$

$\frac{1}{x_1+y_1}$	$\frac{1}{y_1}$
$\frac{1}{x_1}$	1

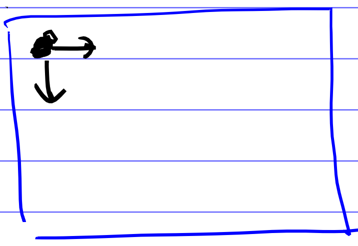
$$\frac{1}{x_1+y_1} \cdot \frac{1}{x_1} \cdot 1 + \frac{1}{x_1+y_1} \cdot \frac{1}{y_1} \cdot 1$$

$$= \frac{1}{x_1 \cdot y_1}.$$

Proof. Induction on $k+\ell$.

Base.

Step of induction.



$$\sum = \sum{}' + \sum{}''$$

the sum over paths starting with a right step

the sum over paths starting with a down step

$$= \frac{1}{x_1+y_1} \cdot \frac{1}{x_2 \dots x_k y_1 \dots y_\ell} +$$

$$+ \frac{1}{x_1+y_1} \cdot \frac{1}{x_1 \dots x_k y_2 \dots y_\ell}$$

$$= \frac{1}{x_1 \dots x_k y_1 \dots y_\ell}. \quad \square$$

Let us now consider the sum over "hook walks" in the rectangle, i.e., paths that can "skip over" rows or columns that end in the lower right corner.

Lemma 2. $$\sum_{\substack{P \text{ is a "hook walk"} \\ \text{in } (\ell+1)\times(k+1) \text{ rectangle}}} wt(P) =$$

$$= \left(1+\frac{1}{x_1}\right)\left(1+\frac{1}{x_2}\right)\cdots\left(1+\frac{1}{x_k}\right) \cdot \left(1+\frac{1}{y_1}\right)\left(1+\frac{1}{y_2}\right)\cdots\left(1+\frac{1}{y_\ell}\right)$$

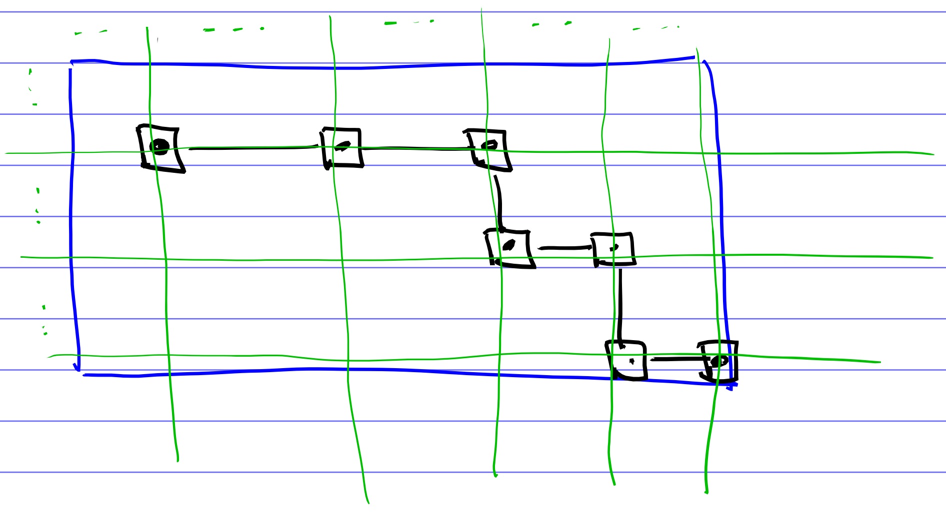
Example $k=\ell=1$

$\frac{1}{x_1+y_1}$	$\frac{1}{y_1}$
$\frac{1}{x_1}$	1

$$\frac{1}{x_1+y_1}\cdot\frac{1}{x_1}\cdot 1 + \frac{1}{x_1+y_1}\cdot\frac{1}{y_1}\cdot 1 + \frac{1}{x_1}\cdot 1 + \frac{1}{y_1}\cdot 1 + 1$$

$$= \left(1+\frac{1}{x_1}\right)\left(1+\frac{1}{y_1}\right).$$

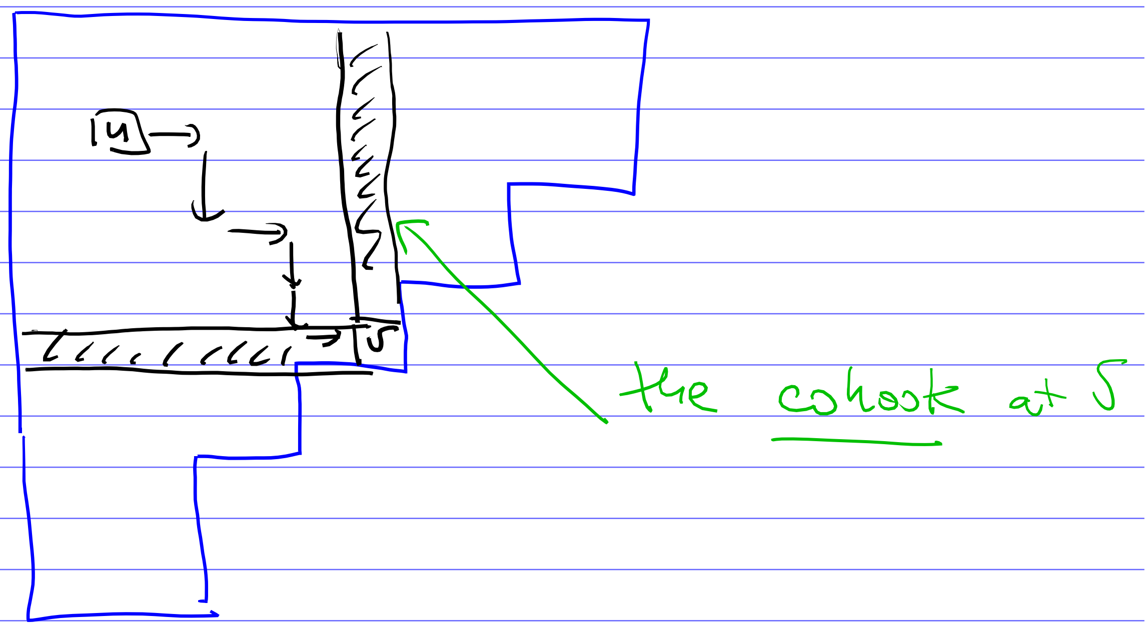
Proof. Any "hook walk" is a "lattice path" for some subrectangle obtained from the $(\ell+1)\times(k+1)$ rectangle by removing some (possibly empty) subsets of first k columns and first ℓ rows.



Each term in the expansion of $\left(1+\frac{1}{x_1}\right)\cdots\left(1+\frac{1}{x_k}\right)\left(1+\frac{1}{y_1}\right)\cdots\left(1+\frac{1}{y_\ell}\right)$ corresponds to a choice of such "subrectangle". Now Lemma 2 follows from Lemma 1. $\square$

Back to hook walks in λ with weights of boxes given by

$$\frac{1}{h(a)-1}.$$



$$P(\delta) := \frac{1}{n} \sum_{u} P(u,\delta) =$$

$$\underset{\text{Lemma 2}}{=} \frac{1}{n} \cdot \prod_{\substack{a \in \mathrm{cohook}(\delta) \\ a \neq \delta}} \left(1 + \frac{1}{h(a)-1}\right)$$

$$= \frac{1}{n} \cdot \prod_{\substack{a \in \mathrm{cohook}(\delta) \\ a \neq \delta}} \frac{h(a)}{h(a)-1}$$

$$= \frac{1}{n} \cdot \frac{H(\lambda)}{H(\lambda \setminus \{\delta\})}$$

Indeed, all hook lengths in λ and $\mu = \lambda \setminus \{\delta\}$ are the same, except the hook lengths of boxes $a \in \mathrm{cohook}(\delta)$. The latter are decreased by 1 in μ.

We obtain

$$\sum_{\substack{s \text{ corner box} \\ \text{in } \lambda}} P(s) = 1$$

$$\Updownarrow$$

$$\frac{1}{n} \sum_{\mu <\cdot \lambda} \frac{H(\lambda)}{H(\mu)} = 1$$

$$\Updownarrow$$

$$\frac{n!}{H(\lambda)} = \sum_{\mu <\cdot \lambda} \frac{(n-1)!}{H(\mu)}$$

This is exactly the needed identity for the recurrence in the RHS of the hook length formula. Q.E.D.

Versions of hook length formulas

Shifted shapes

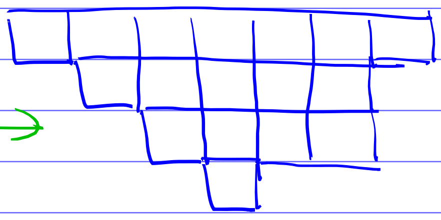
Let $\lambda = (\lambda_1 > \lambda_2 > \ldots > \lambda_\ell)$ be a strict partition of n.

The shifted Young diagram of shape λ is the collection of boxes on the plane s.t.

- the i^{th} row consists of λ_i boxes
- first boxes in rows are "diagonally justified" as shown below

Example

the left side should be a staircase



the shifted Young diagram of shape $\lambda = (7, 5, 4, 1)$

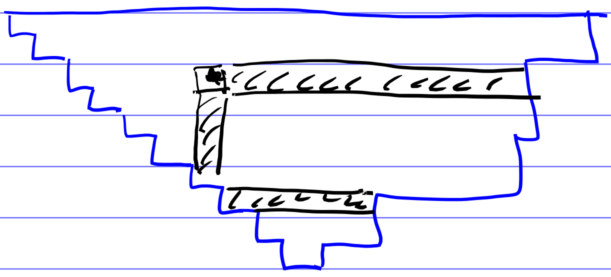
Definition A standard Young tableau of shifted shape λ is a filling of boxes of this shifted shape by $1, 2, \ldots, n$ (w/o repeated entries) which increase in rows & columns.

Example

1	2	3	5	9	10	15
	4	6	8	12	13	
		7	11	16	17	
			14			

an SYT of shifted shape

Hooks with a "broken leg" in a shifted shape



hook lenght $h(a) :=$ # boxes in such "hook with a broken leg" at a

Ex.

12	11	9	7	5	4	1
	9	6	5	3	2	
		5	4	2	1	
			1			

hook lengths of this shifted shape

Shifted Hooklength formula

SYT's of shifted shape λ

$$= \frac{n!}{\prod_{a \in \lambda} h(a)} \qquad \left(n = \# \text{ boxes in } \lambda \right)$$

hook lengths in the shifted shape λ.

Example $\lambda =$

5	3	2
	2	1

hook lengths

shifted SYT's =

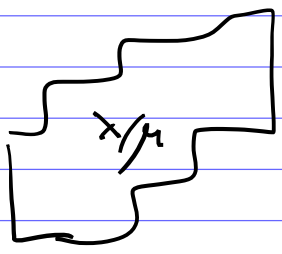
$$= \frac{5!}{5 \cdot 3 \cdot 2 \cdot 2 \cdot 1} = 2.$$

1	2	4
	3	5

1	2	3
	4	5

Exercise Prove the shifted hook length formula. (i.e. modify the probabilistic "hook walk" proof).

Skew shapes λ/μ



should not be confused with shifted shapes

In general,
SYT's of a skew shape λ/μ is not given by a simple product. (This number might involve large prime factors, which are much larger that parts of λ & μ).

So for a long time people thought that there is no way to generalize the hook length formula to skew shapes....

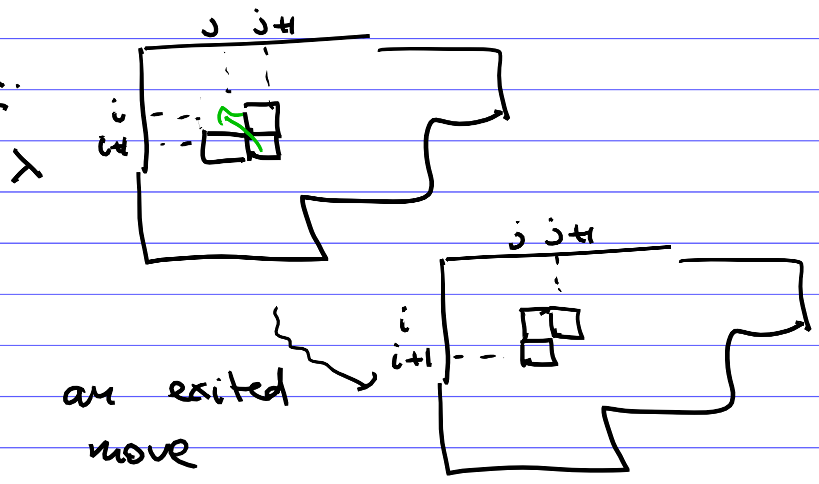
But Naruse '2014 proved the following hook length formula for skew shapes.

Let D be any subset of boxes of λ.

Def Exited Moves

If $(i+1, j+1), (i+1, j), (i, j+1) \in D$, then we can replace the box $(i+1, j+1)$ by (i, j)

Ex.



Def. For a skew shape λ/μ, an exited diagram is any subset of boxes of λ obtained from λ/μ by any number of excited moves.

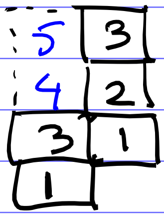
Naruse Hook length formula

Let λ/μ be a skew shape with $|\lambda/\mu| = n$ boxes.

$$f_{\lambda/\mu} = n!\left(\sum_{\substack{D \text{ excited} \\ \text{diagram for} \\ \lambda/\mu}} \prod_{a\in D} \frac{1}{h(a)}\right)$$

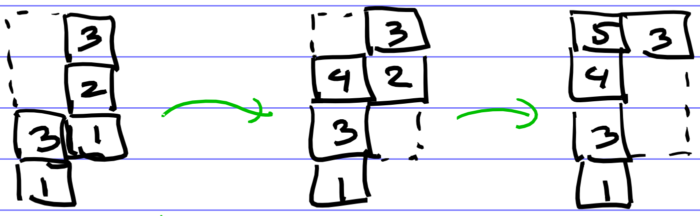
these are usual hook lengths for shape λ

Example

$\lambda/\mu =$ 

hook length for all boxes of λ (including the missing boxes in λ/μ)

Exited diagrams:



the original skew shape

$$f_{2221/11} =$$

$$= 5!\left(\frac{1}{3\cdot3\cdot2\cdot1\cdot1} + \frac{1}{4\cdot3\cdot3\cdot2\cdot1} + \frac{1}{5\cdot4\cdot3\cdot3\cdot1}\right)$$

$$= 9.$$