18.217 Lecture 33 11/20/2020

last lecture: ...

$\forall$ honeycomb with boundary rays given by λ, μ, ν

$$\lambda_1 + \mu_1 \geq \nu_1.$$

Exercise Prove Weyl's inequality using honeycombs. More precisely, prove the following claim:

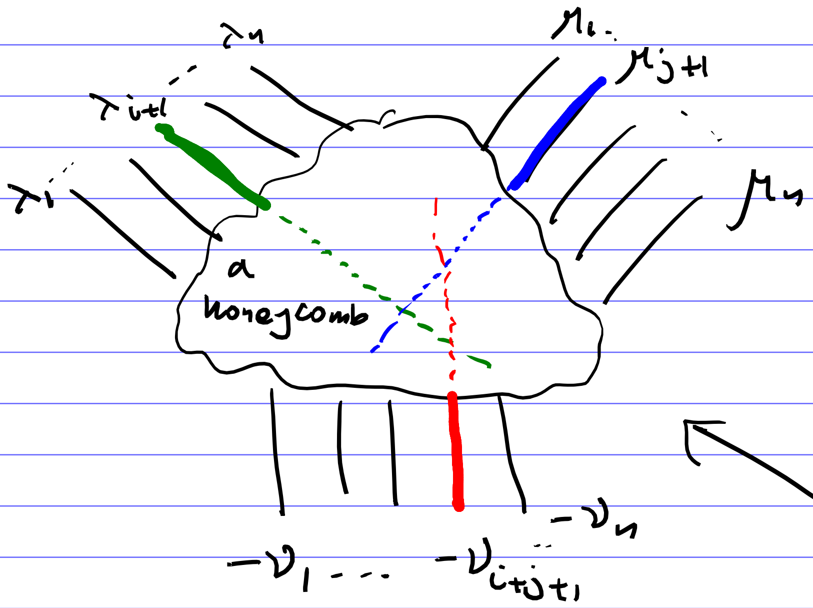
Claim If $c_{\lambda\mu}^{\nu} \neq 0$, then $\lambda_{i+1} + \mu_{j+1} \geq \nu_{i+j+1}$ for any $i, j \geq 0$.

Moreover, for any honeycomb with boundary rays given by λ, μ, ν we have

$$\lambda_{i+1} + \mu_{j+1} \geq \nu_{i+j+1}$$

for $i, j \geq 0$, $i+j < n$

as shown here



You need to show that these 3 lines are arranged like this

Let's us now give <u>all</u> <u>inequalities</u> defining the Klyachko cone.

$\mathrm{Klyachko}(n) :=$

$:= \{ (\lambda, \mu, \nu) \in \mathbb{R}^{3n} \mid$ $\exists$ Hermitian $n\times n$ matrices $A + B = C$ with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$; $\mu_1 \geq \ldots \geq \mu_n$; $\nu_1 \geq \ldots \geq \nu_n$, resp. $\}$

$= \{ (\lambda, \mu, \nu) \in \mathbb{R}^{3n} \mid$ $\exists$ a honeycomb with boundary rays given by parts of $\lambda, \mu, \nu \}$

According to the saturation thm

$\mathrm{Klyachko}(n) \cap \mathbb{Z}^{3n} =$

$= \{ (\lambda, \mu, \nu) \in \mathbb{Z}^{3n} \mid c_{\lambda\mu}^{\nu} \neq 0 \}$

<u>Theorem</u> (Horn conjecture)

The Klyachko cone $\mathrm{Klyachko}(n) \subset \mathbb{R}^{3n}$ is given as follows:

$(\lambda, \mu, \nu) \in \mathrm{Klyachko}(n)$ iff

- $\sum_i \lambda_i + \sum_i \mu_i = \sum_i \nu_i$
- $\lambda_1 \geq \ldots \geq \lambda_n$
  $\mu_1 \geq \ldots \geq \mu_n$
  $\nu_1 \geq \ldots \geq \nu_n$
- $\boxed{\sum_{i\in I} \lambda_i + \sum_{j\in J} \mu_j \geq \sum_{k\in K} \nu_k}$

for any triple of subsets $I, J, K \subset [n]$, $\# I = \# J = \# K = r$, $r \in \{1, \ldots, n-1\}$ such that the Littlewood-Richardson coefficient $c_{\gamma(I), \gamma(J)}^{\gamma(K)} \neq 0$

where $\gamma(I) := (i_r - r, i_{r-1} - (r-1), \ldots, i_1 - 1)$ for $I = \{ i_1 < \ldots < i_r \}$

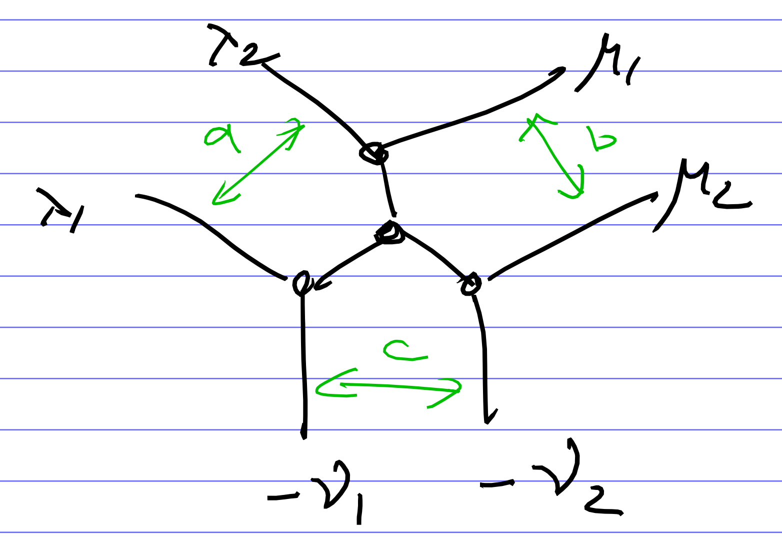
the partition associated with subset I.

This was conjectured by Horn proved by Klyachko (for Hermitian matrices).

Knutson-Tao proved that integer points correspond to $c_{\lambda\mu}^{\nu} \neq 0$.

Example. Klyachko (2) is the cone of all $(\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2) \in \mathbb{R}^6$ s.t.

- $\lambda_1 + \lambda_2 + \mu_1 + \mu_2 = \nu_1 + \nu_2$
- $\lambda_1 \geq \lambda_2, \quad \mu_1 \geq \mu_2, \quad \nu_1 \geq \nu_2$
- $\lambda_1 + \mu_1 \geq \nu_1,$
  $\lambda_1 + \mu_2 \geq \nu_2$
  $\lambda_2 + \mu_1 \geq \nu_2$



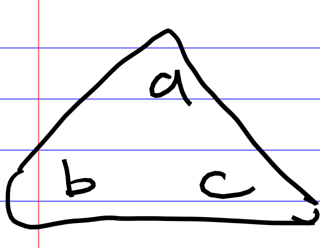
These 3 inequalities correspond to

$c^{\varnothing}_{\varnothing\varnothing} = 1 \qquad (S_\varnothing \cdot S_\varnothing = S_\varnothing)$

$c^{\square}_{\varnothing\square} = 1 \qquad (S_\varnothing \cdot S_\square = S_\square)$

$c^{\square}_{\square\varnothing} = 1 \qquad (S_\square \cdot S_\varnothing = S_\square)$

A related claim:



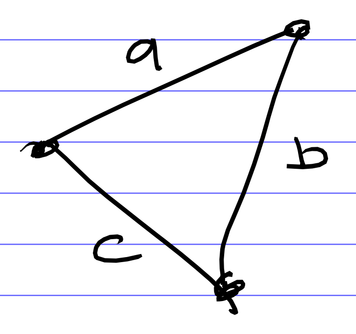
is a BZ-triangle iff a, b, c satisfy the triangle inequalities

$$\begin{cases} a + b \geq c \\ a + c \geq b \\ b + c \geq a \end{cases}$$

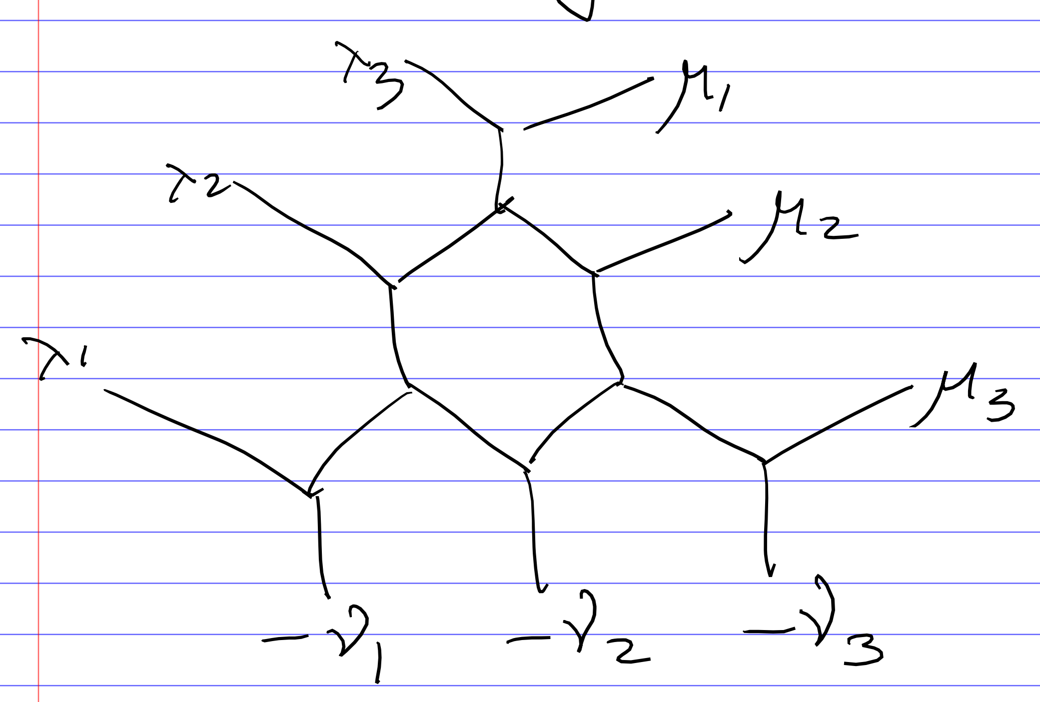
$\Updownarrow$

a usual Euclidean triangle

$\exists$ a triangle with sides of lengths a, b, c:



$n = 3$. We want to describe all triples λ, μ, ν s.t. $\exists$ a honeycomb:



Klyachko (3) is the cone of $(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3) \in \mathbb{R}^9$

- $\sum \lambda_i + \sum \mu_i = \sum \nu_i$
- $\lambda_1 \geq \lambda_2 \geq \lambda_3, \mu_1 \geq \mu_2 \geq \mu_3, \nu_1 \geq \nu_2 \geq \nu_3$
- need to find all non-zero LR-coeffs for partitions that fit inside the 1×2 or 2×1 rectangles.

1×2 rectangle $\square\square$

$S_0 \cdot S_0 = S_0 \rightsquigarrow \lambda_1 + \mu_1 \geq \nu_1$

$S_0 \cdot S_1 = S_1 \rightsquigarrow \lambda_1 + \mu_2 \geq \nu_2$

$S_0 \cdot S_2 = S_2 \rightsquigarrow \lambda_1 + \mu_3 \geq \nu_3$

for S.F.s in 1 variab.

$S_1 \cdot S_0 = S_1 \rightsquigarrow \lambda_2 + \mu_1 \geq \nu_2$

$S_1 \cdot S_1 = S_2 \rightsquigarrow \lambda_2 + \mu_2 \geq \nu_3$

$S_2 \cdot S_0 = S_2 \rightsquigarrow \lambda_3 + \mu_1 \geq \nu_3$

2×1 rectangle $\begin{smallmatrix}\square\\\square\end{smallmatrix}$

$S_{00} \cdot S_{00} = S_{00} \rightsquigarrow \lambda_1 + \lambda_2 + \mu_1 + \mu_2 \geq \nu_1 + \nu_2$

$S_{00} \cdot S_{10} = S_{10} \rightsquigarrow \lambda_1 + \lambda_2 + \mu_1 + \mu_3 \geq \nu_1 + \nu_3$

$S_{00} \cdot S_{11} = S_{11} \rightsquigarrow \lambda_1 + \lambda_2 + \mu_2 + \mu_3 \geq \nu_2 + \nu_3$

$S_{10} \cdot S_{00} = S_{10} \rightsquigarrow \lambda_1 + \lambda_3 + \mu_1 + \mu_2 \geq \nu_1 + \nu_3$

rest. to part. inside 2×1

$S_{10} \cdot S_{10} = S_{11} \rightsquigarrow \lambda_1 + \lambda_3 + \mu_1 + \mu_3 \geq \nu_2 + \nu_3$

$S_{11} \cdot S_{00} = S_{11} \rightsquigarrow \lambda_2 + \lambda_3 + \mu_1 + \mu_2 \geq \nu_2 + \nu_3$

The thm. gives answers to the following 2 questions:

- $\exists$ Hermitian $n \times n$ matrices $A + B = C$ with eigenvalues $(\lambda_1, \dots \lambda_n)$, $(\mu_1 \dots \mu_n)$ $(\nu_1 \dots \nu_n)$ iff $(\lambda, \mu, \nu) \in \text{Klyachko}(n)$

- $c_{\lambda\mu}^{\nu} \neq 0$ iff $(\lambda, \mu, \nu) \in \text{Klyachko}(n) \cap \mathbb{Z}^{3 \cdot n}$.

Horn conjectured these inequalities for eigenvalues of Hermitian matrices.
Klyachko proved that they are necessary & sufficient.
Knutson-Tao proved that this describes all $c_{\lambda\mu}^{\nu} \neq 0$.

Thm says that non-zero LR-coefficients are described in terms of non-zero LR-coeffs. ???

For any partitions λ, μ, ν with $\leq n$ parts (that can be arbitrarily large) in order to figure out whether $c_{\lambda\mu}^{\nu} \neq 0$ we need to know whether $c_{\tilde{\lambda}\tilde{\mu}}^{\tilde{\nu}} \neq 0$ for $\tilde{\lambda}, \tilde{\mu}, \tilde{\nu} \subseteq r \times (n-r)$ for some $r \in \{1, .., n-1\}$.

There are finitely many such $\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}$'s and all of them have $\lneq n$ parts.

for given n

So Theorem gives a recursive description of triples of partitions λ, μ, ν with $c_{\lambda\mu}^{\nu} \neq 0$.

Open Problem. Is there a non-recursive description of such triples of partitions?

An application of honeycombs:

The PRV conjecture

(after Parthasarathy, Ranga Rao, Varadarajan)

Proved by Kumar and Mathieu.

For type A.

<u>Theorem</u> Let

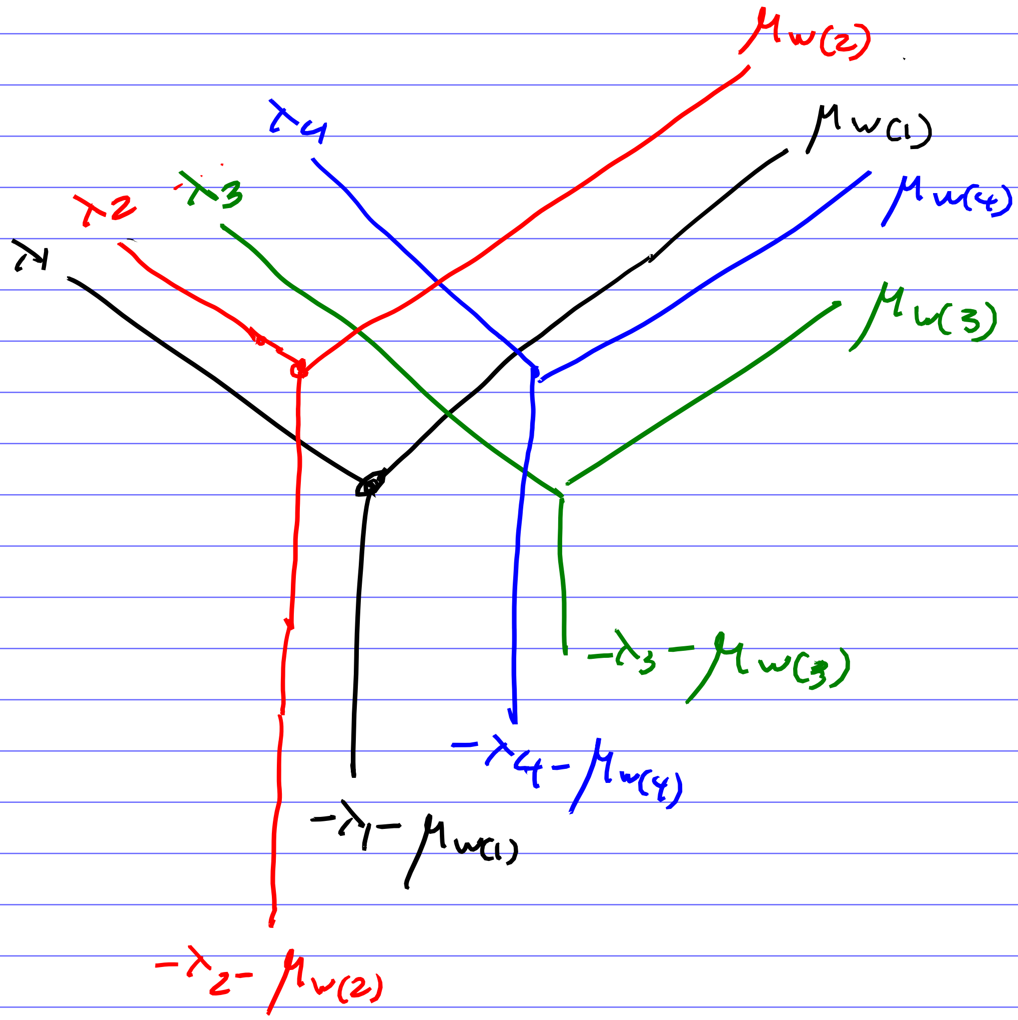
$\lambda = (\lambda_1 \dots \lambda_n)$, $\mu = (\mu_1 \dots \mu_n)$ be two partitions, and $w \in S_n$

ν - weakly decreasing rearrangement of parts of $(\lambda_1 + \mu_{w(1)}, \lambda_2 + \mu_{w(2)}, \dots, \lambda_n + \mu_{w(n)})$.

Then $c_{\lambda\mu}^{\nu} \neq 0$.

Let's explain PRV conjecture using honeycombs:

<u>Proof</u>. Consider the honeycomb obtained by a union of several "Y"s.



This is an integer honeycomb.

So $c_{\lambda\mu}^{\nu} \neq 0$. $\square$

Special case:
$\lambda = \mu = (n, n-1, \ldots, 1)$

In this case, $(\mu_{w(1)}, \ldots, \mu_{w(n)})$ is a permutation of $1, \ldots, n$ (viewed as a vector in $\mathbb{R}^n$).

PRV vectors = "sums of 2 permutations"
$1+w(1), 2+w(2), \ldots, n+w(n)$
(rearranged in decreasing order)
for all $w \in S_n$

Problem Give a combinatorial description of PRV vectors.

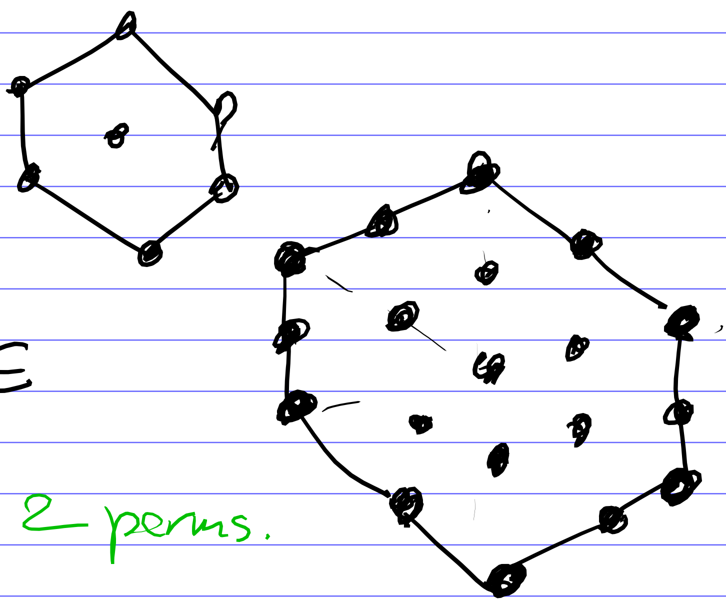
For a permutation $w = w_1 \ldots w_n$
$(w_1, \ldots, w_n) \in$ the permutohedron
$\Pi_n := \mathrm{Conv}\left(\begin{matrix}\text{such} \\ \text{perm.} \\ \text{vectors}\end{matrix}\right)$

So a sum of two permutations $\in 2\Pi_n$

$w \in$

$u + w \in$

a sum of 2 perms.



we get.

Proposition. If $(\delta_1, \ldots, \delta_n) \in \mathbb{Z}^n$ is a PRV vector (i.e. it is a sum of 2 permutations) then

- $\delta_1 \geq \ldots \geq \delta_n$
- $\delta \in 2\Pi_n$

$\Updownarrow$

$$\begin{cases} \delta_1 \leq 2n \\ \delta_1 + \delta_2 \leq 2(n + (n-1)) \\ \delta_1 + \delta_2 + \delta_3 \leq 2(n + (n-1) + (n-2)) \\ \quad \ldots \\ \delta_1 + \delta_2 + \ldots + \delta_{n-1} \leq 2(n + (n-1) + \ldots + 2) \\ \delta_1 + \delta_2 + \ldots + \delta_n = 2(n + \ldots + 1) \end{cases}$$

These are necessary conditions for PRV vectors.

Q: Are they sufficient?

A. No.

Clearly, a permutation (viewed as a vector in $\mathbb{R}^n$) is

$(n+1, n+1, \ldots, n+1)$ − a permutation.

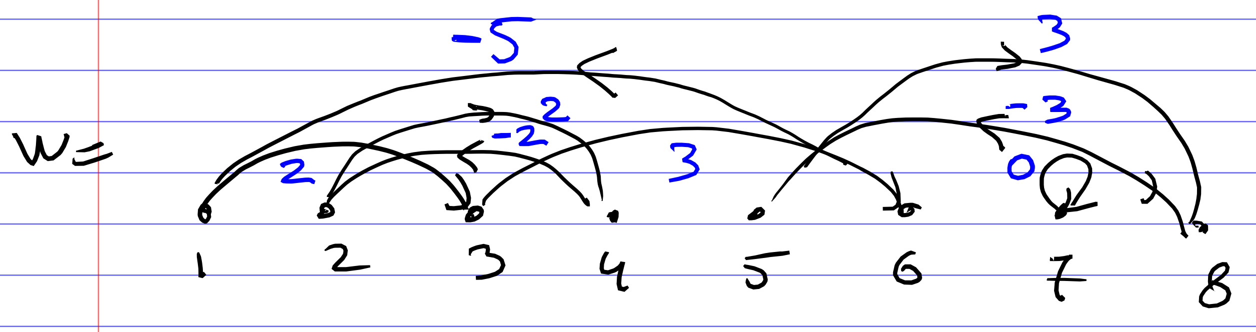
So we've got an equivalent problem:

<u>Problem</u> Describe all possible <u>differences</u> of 2 permutations, i.e. all possible vectors obtained by rearrangement of parts of

$$(w_1 - 1, w_2 - 2, w_3 - 3, \ldots, w_n - n)$$

Example

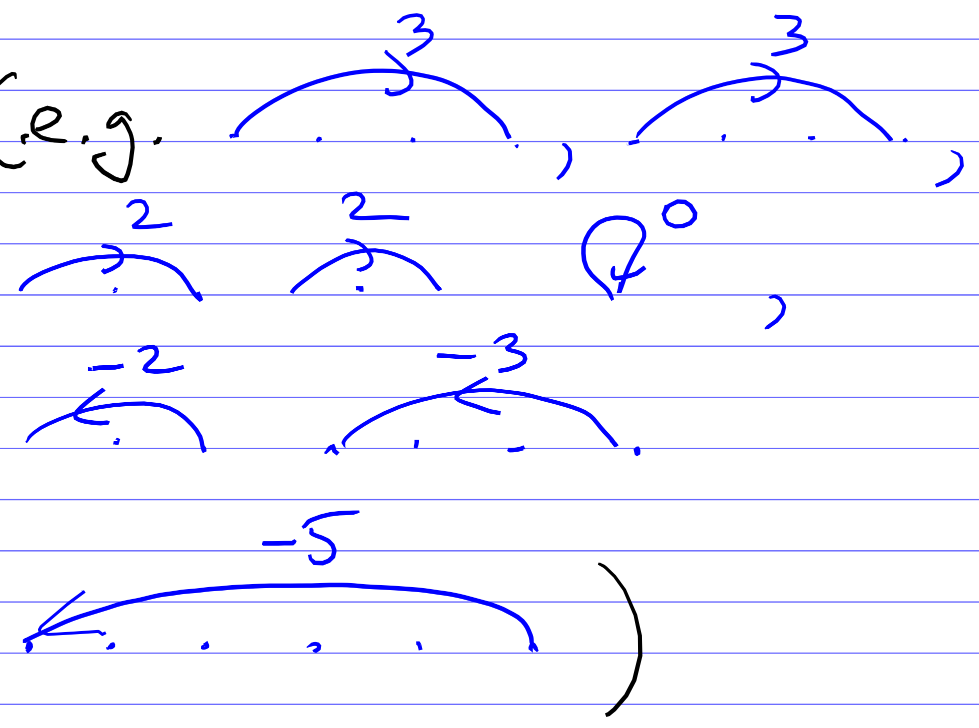
$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 6 & 2 & 8 & 1 & 7 & 5 \end{pmatrix}$$



$(3, 3, 2, 2, 0, -2, -3, -5)$

Problem ⟺ Given a collection of "directed arcs"

(e.g.



)

Can we combine these arcs and get a valid permutation.

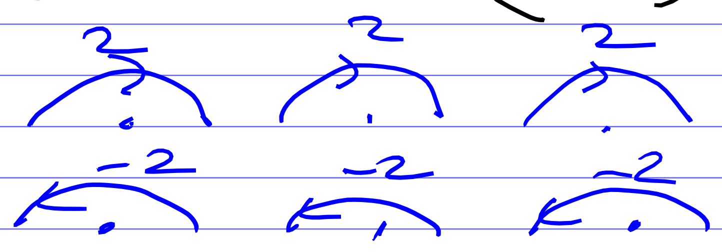
Example of a vector that satisfies the necessary conditions for a diff. of two permutations.

- $\delta_1 \geq \delta_2 \geq \dots \geq \delta_n$
- $$\begin{cases} \delta_1 \leq n-1 \\ \delta_1 + \delta_2 \leq n + (n-1) - (1+2) \\ \delta_1 + \delta_2 + \delta_3 \leq n + (n-1) + (n-2) - (1+2+3) \\ \dots \\ \delta_1 + \dots + \delta_n = 0 \end{cases}$$

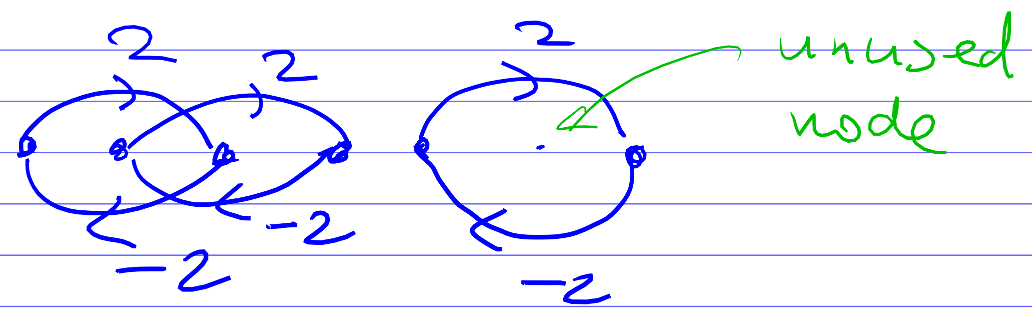
which is not a difference of two permutations:

$n = 6$

$(\delta_1, \dots, \delta_6) = (2, 2, 2, -2, -2, -2)$



We cannot combine those arcs & get a valid permutation:



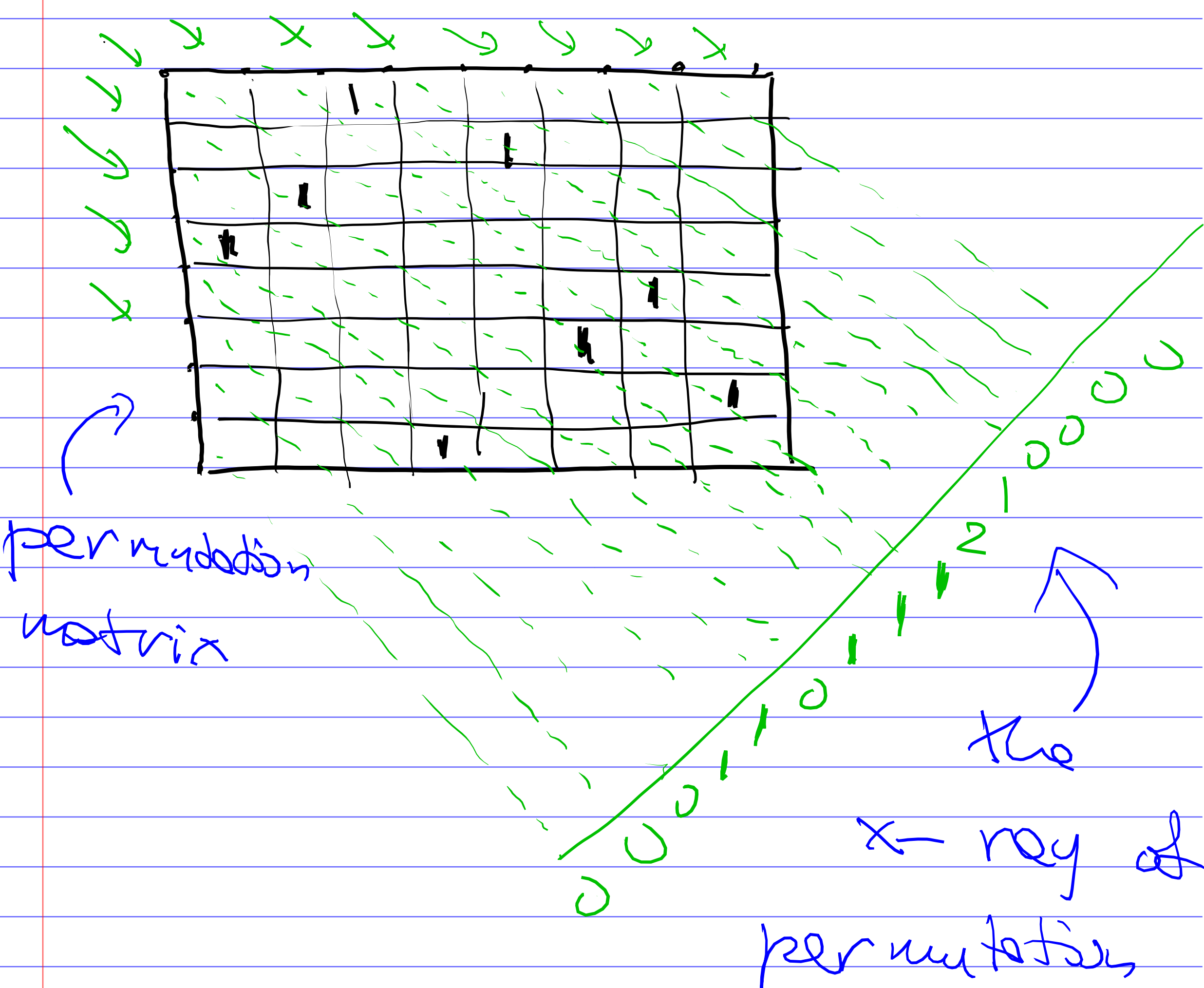
Conjecture The conditions are sufficient if we require the strict inequalities $\delta_1 > \delta_2 > \dots > \delta_n$

A related problem:

Describe

X-rays of permutations

Ex. $n=8$



More on the Littlewood-Richardson coefficients ...

3 ways to relate the LR-coefficients to the Schur functions:

- $s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$
- $s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu$
- $s_\lambda(x_1, x_2, \ldots, y_1, y_2, \ldots) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(x_1 x_2 \ldots) s_\nu(y_1 y_2 \ldots).$

Exercise Prove this identity.

This is related to the fact that the ring of symmetric functions Λ has the structure of self-dual Hopf algebra.

In fact, Zelevinsky proved that Λ is the only commutative self-dual Hopf algebra.