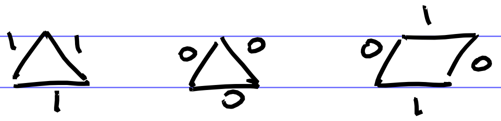
18.217 Lecture 31 11/16/2020

BZ triangles, honeycombs, & puzzles ...

$\lambda = (\lambda_1, \ldots, \lambda_k)$, $\mu = (\mu_1, \ldots, \mu_k)$, $\nu = (\nu_1, \ldots, \nu_k)$

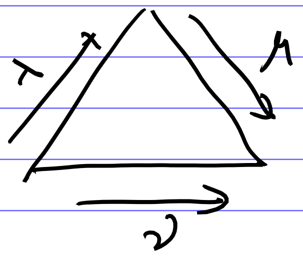
3 partitions $\subseteq k \times (n-k)$

A Knutson–Tao **puzzle** is a tiling of the "large triangle" with sides of length n by the **puzzle pieces**:



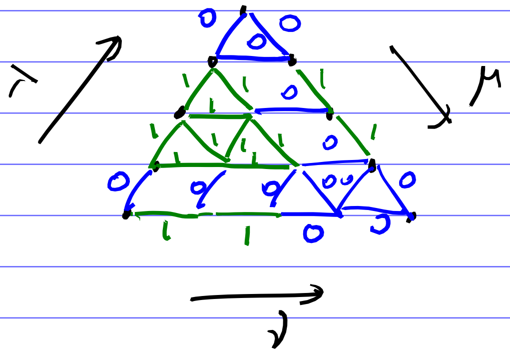
(& rotations)

s.t. the labels of puzzle pieces match on their common sides & λ, μ, ν (encoded by 01-vectors) appear on the sides of the large triangle:

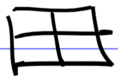


Example

$n=4$, $k=2$



$\lambda = \mu =$ 

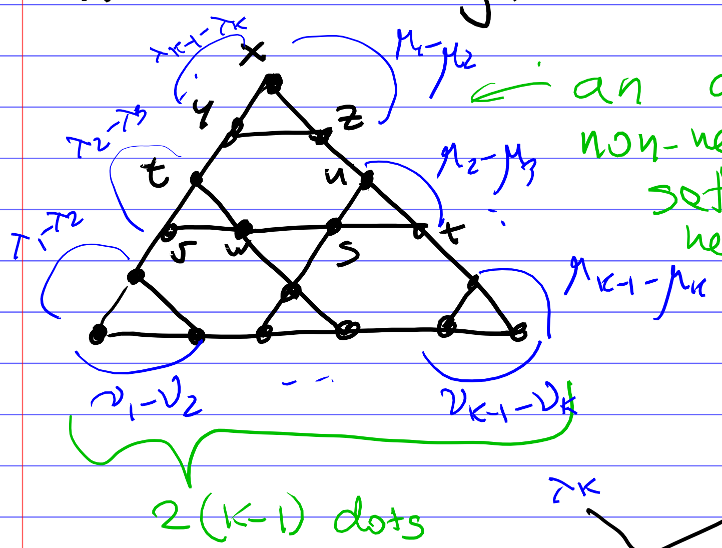
$\nu =$ 

$$\left\{\begin{array}{c}\text{integer BZ-triangles} \\ \text{for } \lambda, \mu, \nu\end{array}\right\}$$

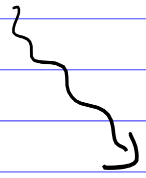
bij. ↕   ↕ bij.

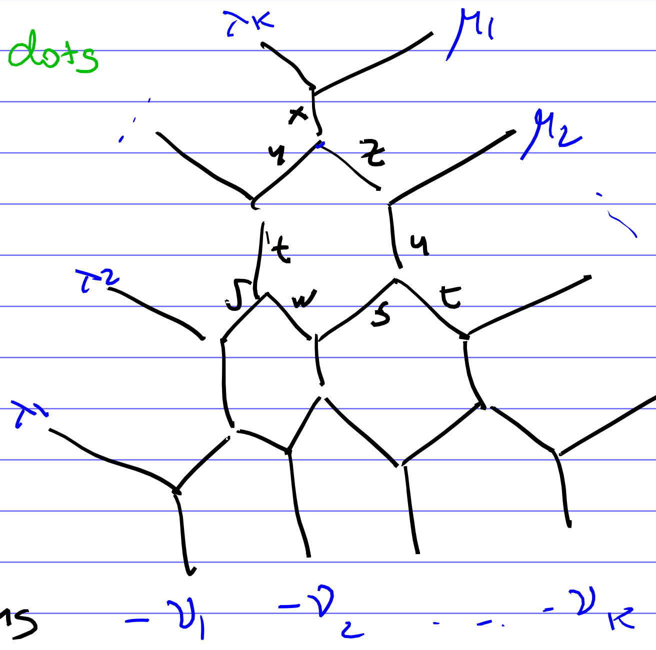
$$\left\{\begin{array}{c}\text{integer honeycombs} \\ \text{for } \lambda, \mu, \nu\end{array}\right\} \qquad \left\{\begin{array}{c}\text{puzzles for} \\ \lambda, \mu, \nu\end{array}\right\}$$

A BZ-triangle



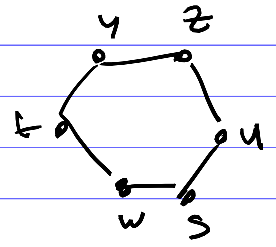
← an array of non-negative integers satisfying the hexagon condition



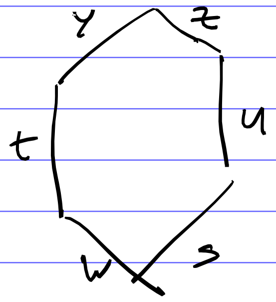


honeycomb with boundary rays given by λ, μ, ν and the lengths of all edges given by the entries of the BZ-triangle.

hex. condition $\Longleftrightarrow$ such honeycomb exists

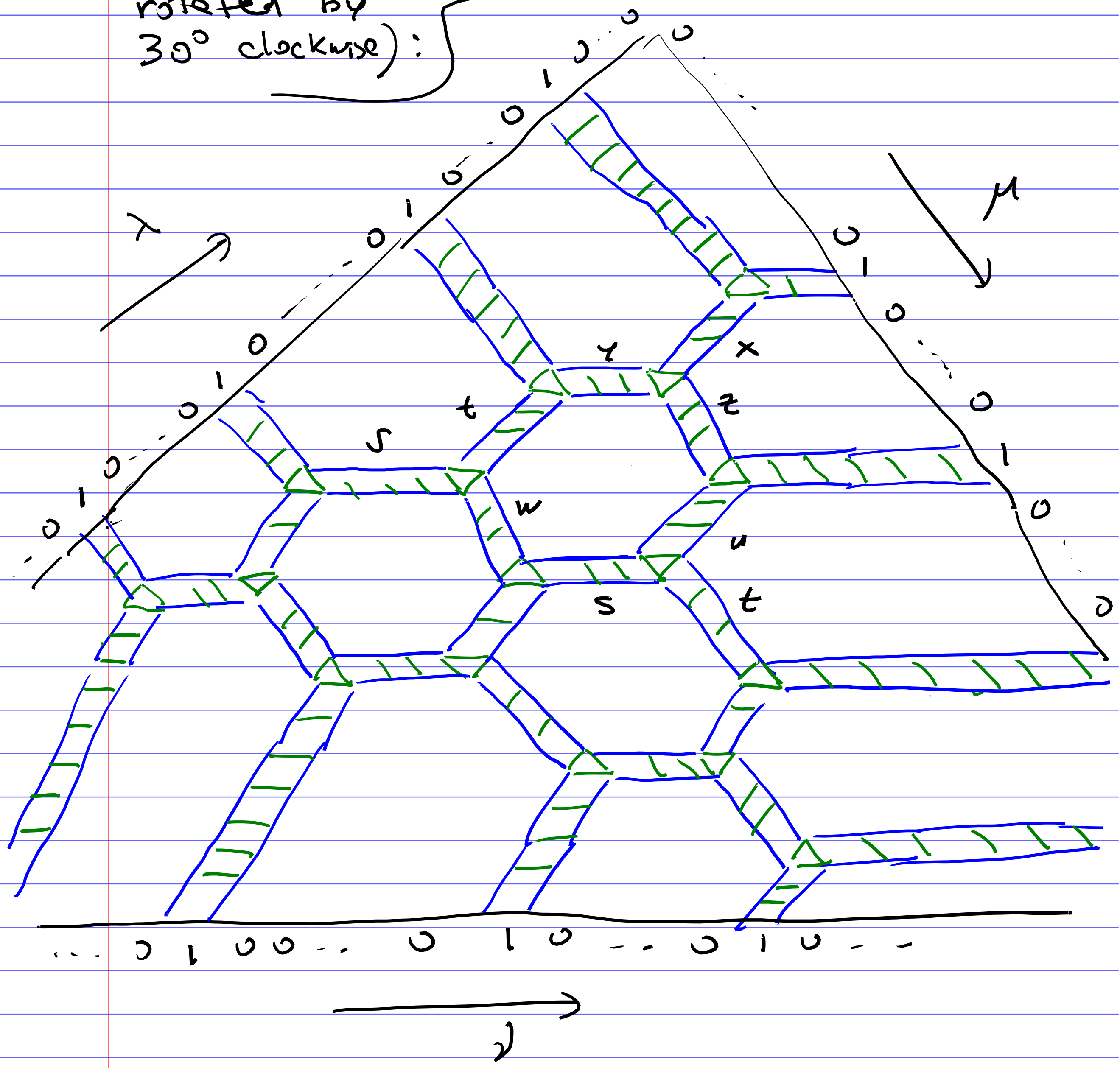


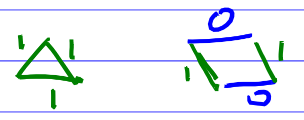
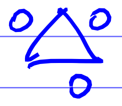
$\exists$ hexagon (with 120° angles) & sides of lengths



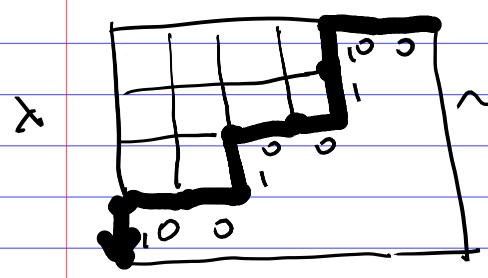
$$\left.\begin{array}{l} y+z = s+w \\ z+u = w+t \end{array}\right\} \Leftarrow$$

The BZ-triangle also corresponds to the following puzzle (which is the "ribbonized honeycomb" rotated by 30° clockwise):



where  and the areas between the ribbons are filled with triangles 

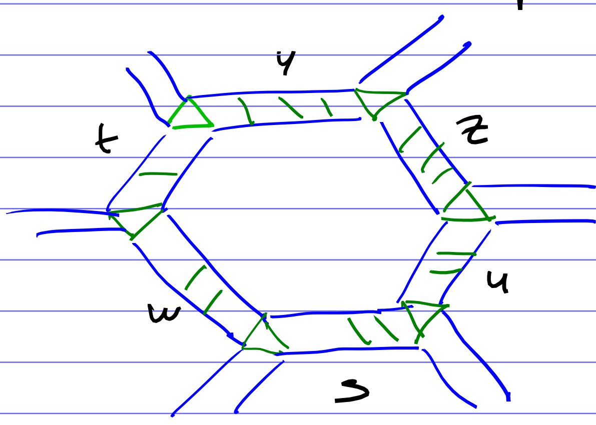
- the "lengths" of edges (= # rhombuses) = the entries of the BZ-triangle
- the positions of 1's & 0's sides sides are given by λ, μ, ν



$\rightsquigarrow$ 0 0 1 1 0 0 1 0 0 1

(1's correspond to parts of λ)

Again, the hex. condition $\Longleftrightarrow$ such ribbonized puzzle exists



$y + z = s + w$

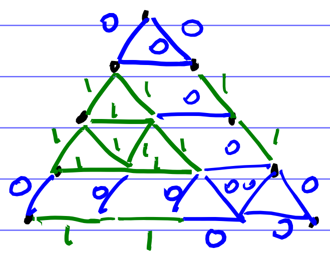
$t + u = w + t$

$u + s = t + y$

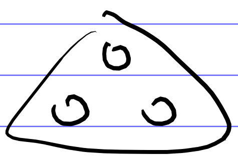
So every BZ-triangle produces a valid puzzle.

<u>Exercise</u>. Prove that every puzzle has this form, i.e. every puzzle comes from a BZ-triangle a shown above.

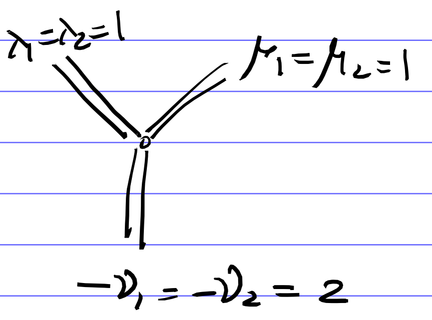
<u>Example</u>:



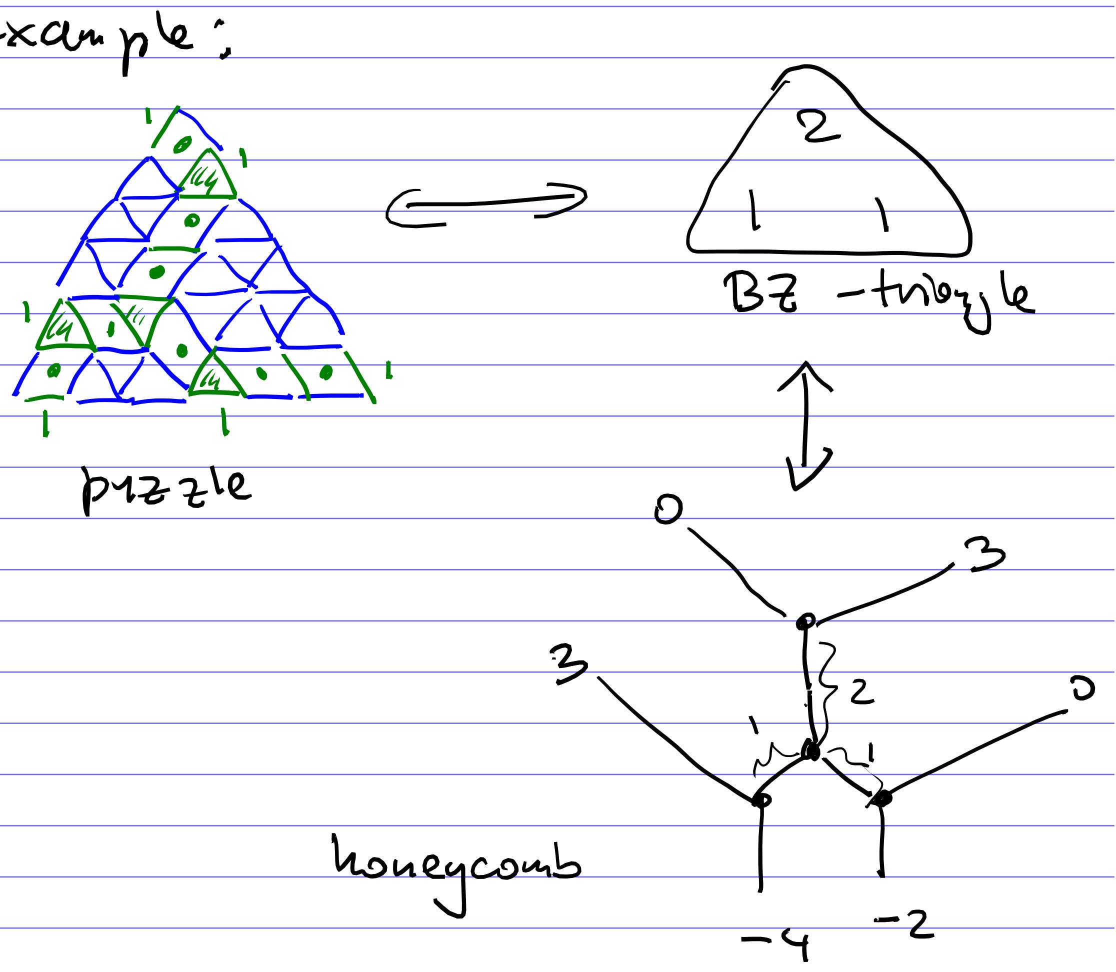
This puzzle corresponds to the BZ-triangle



and the honeycomb:



A more interesting example:

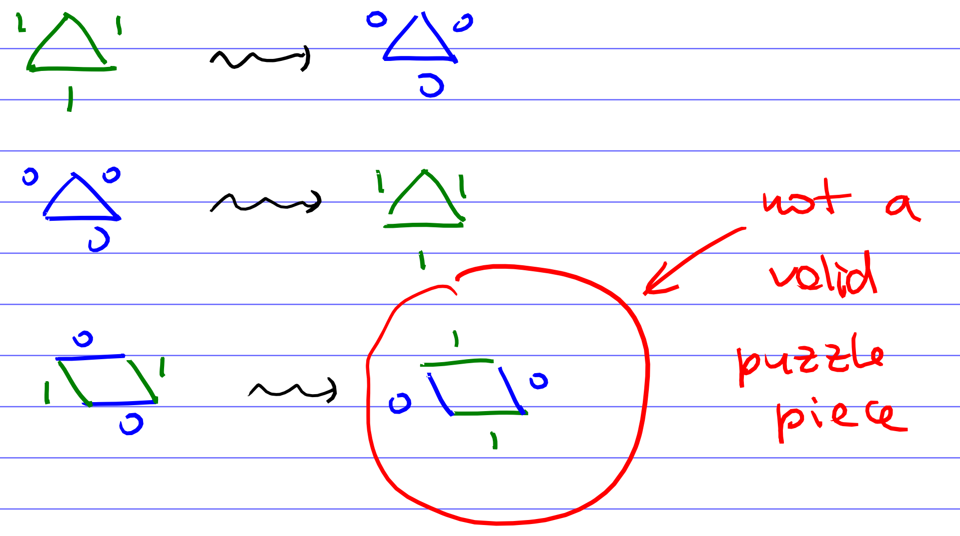


So we have

<u>Theorem</u> $c_{\lambda\mu}^{\nu} = \#$ puzzles for λ, μ, ν.

<u>Symmetries of the Littlewood-Richardson coefficients</u> . . .

If we switch $0 \leftrightarrow 1$ in a puzzle, we will <u>not</u> get a valid puzzle:



But if we switch $0 \leftrightarrow 1$ AND <u>reflect</u>, we get a valid puzzle.

For a puzzle P, let P' be the puzzle obtained from P by switching $0 \leftrightarrow 1$ and reflecting it w.r.t. the vertical axis.

where λ', μ', ν' are the conjugate partitions.

This explains the following symmetry of the LR-coeffs

Theorem $c_{\lambda\mu}^{\nu} = c_{\mu'\lambda'}^{\nu'}$

The puzzle version of LR-rule is the most symmetric formulation of LR-rule

<u>Theorem</u> We obtain the following symmetries of the LR-coeffs by simple transformations of BZ-triangles, honeycombs, & puzzles:

- Cyclic $\mathbb{Z}/3\mathbb{Z}$ symmetry:

$$\tilde{c}_{\lambda\mu\nu} = \tilde{c}_{\mu\nu\lambda} = \tilde{c}_{\nu\lambda\mu}$$

(where $\tilde{c}_{\lambda\mu\nu} = c_{\lambda\mu}^{\nu^*}$)

(Obtained by <u>rotations</u> of BZ-triangles, honeycombs, or puzzles)

- $$c_{\lambda\mu}^{\nu} = c_{\mu^*\lambda^*}^{\nu^*}$$

where $\lambda^* = (-\lambda_n, \dots, -\lambda_1)$

(<u>reflections</u> of BZ-triangles or honeycombs)

- $$c_{\lambda\mu}^{\nu} = c_{\mu'\lambda'}^{\nu'}$$

(switching $0 \leftrightarrow 1$ & reflections of puzzles).

The only symmetry that does not easily follow from any of these rules is

$$c_{\lambda\mu}^{\nu} = c_{\mu\lambda}^{\nu}$$

(the commutation symmetry

$$s_\lambda \cdot s_\mu = s_\mu \cdot s_\lambda)$$

Horn's Problem & Klyachko's Cone.

<u>Horn' Problem:</u> A, B, C are 3 Hermitian $n\times n$ matrices such that

$$A + B = C.$$

Describe all possible collections of eigenvalues of A, B, C.

Recall that a complex matrix $A = (a_{ij})$ is <u>Hermitian</u> if

$$A = A^*, \quad \text{where}$$

$$A^* = \bar{A}^T \quad \text{(transpose \& complex conjugate)}$$

Any Hermitian matrix has <u>real</u> eigenvalues.

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be the vector of eigenvalues of A arranged in the decreasing order

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \quad (\text{Here } \lambda_i \in \mathbb{R})$$

Similarly $\mu = (\mu_1, \dots \mu_n)$ and $\nu = (\nu_1, \dots \nu_n)$ are vectors of eigenvalues of matrices B and C.

<u>Horn's Problem</u> Describe all triples $(\lambda, \mu, \nu) \in \mathbb{R}^{3n}$ such that $\exists$ Hermitian matrices A, B, C with vectors of eigenvalues λ, μ, ν, resp., such that $A + B = C$.

Some necessary conditions:

- $$\underbrace{\sum_i \lambda_i}_{\operatorname{tr}(A)} + \underbrace{\sum_i \mu_i}_{\operatorname{tr}(B)} = \underbrace{\sum_i \nu_i}_{\operatorname{tr}(C)}$$

- $\lambda_1 + \mu_1 \geq \nu_1$,

  i.e. the largest eigenvalue of $A+B$ is $\leq$ sum of the largest eigenvalues of A and B.

More generally,

- H. Weyl: $\lambda_{i+1} + \mu_{j+1} \geq \nu_{i+j+1}$ for $i, j \geq 0$, $i+j < n$.

Notice that all these conditions are given by linear (in)equalities for λ_i's, μ_i's, ν_i's.

Actually, the set of such triples (λ, μ, ν) is a certain polyhedral cone in $\mathbb{R}^{3n}$, whose structure was initially conjectured by Horn and proved by Klyachko.

It is called the Klyachko cone $\mathrm{Klyachko}(n) \subset \mathbb{R}^{3n}$.

A seemingly unrelated problem...

λ, μ, ν are 3 "partitions" (where we allow negative parts) with at most n parts (expressed as vectors in $\mathbb{Z}^n$)

Describe all triples $(\lambda, \mu, \nu) \in \mathbb{Z}^{3n}$ such that

$$c_{\lambda \mu}^{\nu} \neq 0.$$

Equivalently, describe all triples $(\lambda, \mu, \nu) \in \mathbb{Z}^{3n}$ s.t. there exists an integer honeycomb with boundary rays given by λ, μ, ν.

Theorem $c_{\lambda\mu}^{\nu} \neq 0$ iff (λ, μ, ν) is an integer lattice point of the Klyachko cone:
$$(\lambda, \mu, \nu) \in \text{Klyachko}(n) \cap \mathbb{Z}^{3n}.$$

The long story of the proof of this theorem: works of Horn, Klyachko, and others. Klyachko proved this Theorem modulo the saturation conjecture.

The last step of the proof was provided by Knutson–Tao who proved Klyachko's saturation conjecture using honeycombs.

Saturation Theorem [Knutson–Tao] Let λ, μ, ν be 3 partitions, $k \in \mathbb{Z}_{>0}$.

If $c_{k\lambda, k\mu}^{k\nu} \neq 0$, then $c_{\lambda\mu}^{\nu} \neq 0$.

Here $k\lambda := (k\lambda_1, \dots, k\lambda_n)$ for $\lambda = (\lambda_1 \dots \lambda_n)$.

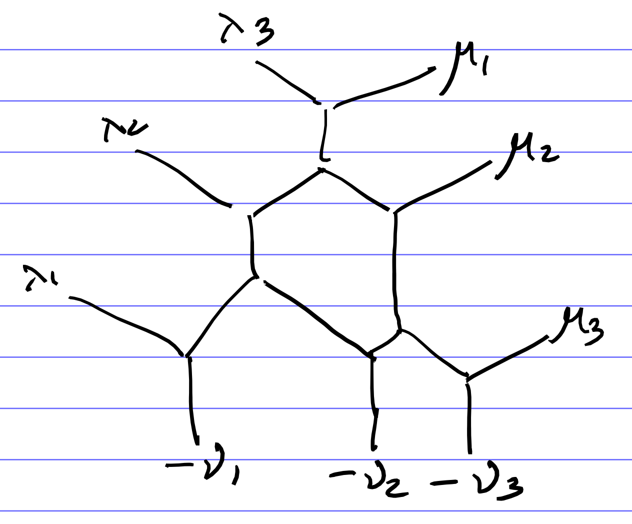
Basically, $(\lambda, \mu, \nu) \subset \text{Klyachko}(n)$ iff there is a (real valued) honeycomb with boundary rays given by λ, μ, ν.

$c_{\lambda\mu}^{\nu} \neq 0$ iff $\exists$ an integer honeycomb with boundary rays given by λ, μ, ν.

Knutson–Tao proved

Theorem For $(\lambda, \mu, \nu) \in \mathbb{Z}^{3n}$ if there is some honeycomb with boundary rays given by λ, μ, ν, then there is an integer honeycomb with the same boundary rays.

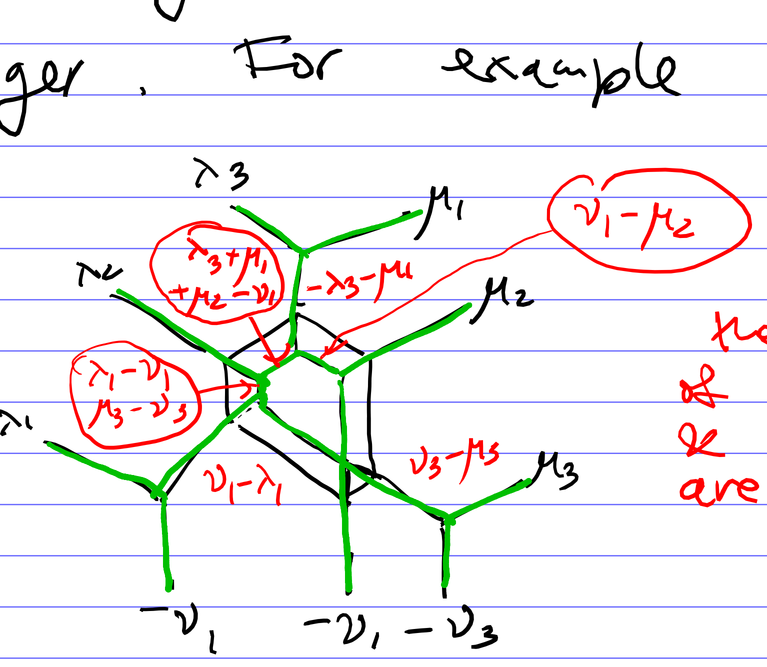
Ex.



This honeycomb may not be integer even if the coord. of all boundary rays are integer

$(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3 \in \mathbb{Z})$

However, by making the hexagon smaller or larger we can deform the honeycomb s.t. it becomes integer. For example



the coord. of all lines & line segments are integer.

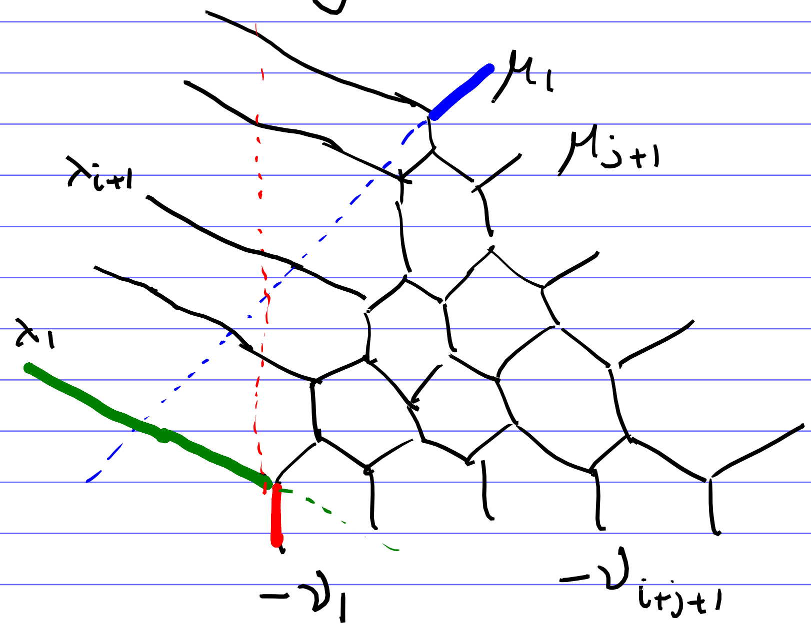
an integer honeycomb with the same boundary rays.

Let's us show some examples that illustrate the relationship between Horn's problem & positivity of the LR-coefficients.

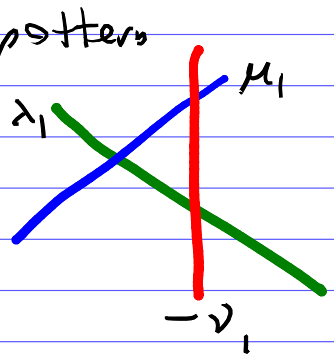
We mentioned some inequalities for Horn's problem

$$\lambda_1 + \mu_1 \le \nu_1$$

Let's explain this in terms of honeycombs

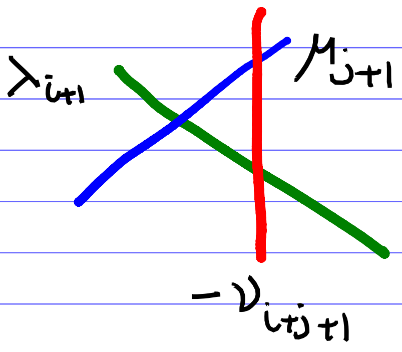


$\lambda_1 + \mu_1 \le \nu_1 \iff$ these 3 lines form the following pattern



<u>Weyl's inequality</u>

$\lambda_{i+1} + \mu_{j+1} \le \nu_{i+j+1} \iff$ the same is true for the lines corresponding to $\lambda_{i+1}, \mu_{j+1}, \nu_{i+j+1}$



We gave some inequalities for the Klyachko's cone.
Let us now describe <u>all</u> inequalities.

<u>Theorem</u> The Klyachko's cone $\mathrm{Klyachko}(n) \subset \mathbb{R}^{3n}$ is given as follows:

$(\lambda, \mu, \nu) \in \mathrm{Klyachko}(n)$ iff

- $\sum_i \lambda_i + \sum_i \mu_i = \sum_i \nu_i$
- $\lambda_1 \geq \ldots \geq \lambda_n$
  $\mu_1 \geq \ldots \geq \mu_n$
  $\nu_1 \geq \ldots \geq \nu_n$
- $$\boxed{\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j \geq \sum_{k \in K} \nu_k}$$

for any triple of subsets $I, J, K \subset [n]$ with $|I| = |J| = |K| = r$, $r = 1, \ldots, n-1$ such that the Littlewood-Richardson coefficient $c^{\gamma(K)}_{\gamma(I), \gamma(J)} \neq 0$

where $\gamma(I) := (i_r - r, i_{r-1} - (r-1), \ldots, i_1 - 1)$ for $I = \{i_1 < i_2 < \ldots < i_r\}$

- $\exists$ Hermitian matrices $A + B = C$ with eigenvalues λ_i, μ_i, ν_i iff $(\lambda, \mu, \nu) \in \mathrm{Klyachko}(n)$
- $c^{\nu}_{\lambda\mu} \neq 0$ iff $(\lambda, \mu, \nu) \in \mathrm{Klyachko}(n)$.

The fact that these inequalities are sufficient was conjectured by Horn (for Hermitian matrices) proved by Klyachko (for Herm. mat) and Knutson-Tao (for $c^{\nu}_{\lambda\mu} \neq 0$)

So this theorem says non-zero LR-coefficients are described in terms of non-zero LR-coefficients.

For any partitions λ, μ, ν with $\leq n$ parts (that can be arbitrarily large), in order to know whether $c_{\lambda\mu}^{\nu} \neq 0$ only need to know whether $c_{\tilde{\lambda}\tilde{\mu}}^{\tilde{\nu}} \neq 0$ for

$\tilde{\lambda}, \tilde{\mu}, \tilde{\nu} \subseteq r \times (n-r)$

for some $r \in \{1, 2, \ldots, n-1\}$.

There are finitely many such $\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}$, and all of them have $\lneq n$ parts.

So we get a <u>recursive</u> description triples of partitions λ, μ, ν with $c_{\lambda\mu}^{\nu} \neq 0$.

<u>Open Problem.</u> Is there a <u>non-recursive</u> description of such triples (λ, μ, ν) with $c_{\lambda\mu}^{\nu} \neq 0$?