18.217 Lecture 2

last time: $\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \ldots$

the ring of symmetric functions

(its elements are power series in infinitely many variables $x_1, x_2, \ldots$ with bounded degrees)

$$e_k := \sum_{i_1 < \ldots < i_k} x_{i_1} \ldots x_{i_k}$$ elementary SFs

$$h_k := \sum_{j_1 \leq \ldots \leq j_k} x_{j_1} \ldots x_{j_k}$$ complete homogeneous SFs

$$p_k := x_1^k + x_2^k + x_3^k + \ldots$$ power SFs

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$

$$e_\lambda := e_{\lambda_1} \cdot e_{\lambda_2} \cdot \ldots \cdot e_{\lambda_\ell}$$

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \ldots h_{\lambda_\ell}$$

$$p_\lambda := p_{\lambda_1} p_{\lambda_2} \ldots p_{\lambda_\ell}$$

$$m_\lambda := \sum_{i_1 \ldots i_\ell \text{ distinct}} x_{i_1}^{\lambda_1} \ldots x_{i_\ell}^{\lambda_\ell}$$ monomial SFs

sum of monomials, all coeffs. = 1

Fundemental Theorem of Sym. Functions

$$\Lambda = \mathbb{Z}[e_1, e_2, \ldots].$$

Equivalently, $\{e_\lambda \mid \lambda \text{ partition}\}$ is a $\mathbb{Z}$-linear basis of Λ.

<u>Note:</u> $e_1, e_2, \ldots$ algebraically indep.
$\iff$ e_λ's are linearly indep.

We proved this by showing that

$\{e_\lambda\} = A\{m_\lambda\}$ for an upper-triangular matrix A with 1's on the diagonal.

So $\{m_\lambda\}$ is a linear basis of Λ
$\iff$ $\{e_\lambda\}$ is a linear basis of Λ.

How about the h-version?

<u>Theorem</u> $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$.
Equivalently, $\{h_\lambda\}$ is a $\mathbb{Z}$-linear basis of Λ.

A similar approach fails:

$h_\lambda = \sum$ <u>all</u> monomials $x_{i_1}^{a_1} \cdots x_{i_r}^{a_r}$
of some degree $|\lambda| = \lambda_1 + \cdots + \lambda_\ell$
with some <u>non-zero</u> coeffs.

So $\{h_\lambda\} = B\,\{m_\lambda\}$

<u>not</u> an upper-triang. matrix

We'll try a different approach

We'll relate e_k's with h_ℓ's.

Generating functions:

$$E(t) := \sum_{k \geq 0} e_k t^k$$

$$H(t) := \sum_{\ell \geq 0} h_\ell t^\ell$$

(by convention, $e_0 = h_0 = p_0 = 1$)

Proposition. $\boxed{E(t) \cdot H(-t) = 1}$

This allows to express e_k's in terms of h_ℓ's and vise versa:

$$E(t) = \frac{1}{H(-t)} = \frac{1}{1 - h_1 t + h_2 t^2 - \ldots}$$

$$= 1 + (h_1 t - h_2 t^2 + \ldots) + (h_1 t - h_2 t^2 + \ldots)^2 + \ldots$$

So $e_1 = h_1$,

$e_2 = -h_2 + h_1^2$,

$e_3 = h_3 - 2h_2h_1 + h_1^3$, etc.

<u>Note</u>. Since the relation between $E(t)$ & $H(t)$ is <u>invariant</u> w.r.t. switching E & H, exactly the <u>same relations</u> hold for expressions of h_k's in terms of e_ℓ's:

$h_1 = e_1$, $h_2 = -e_2 + e_1^2$,

$h_3 = e_3 - 2e_2e_1 + e_1^3$, etc.

Thus $\{e_\lambda\}$ linear basis of Λ

$\iff$ $\{h_\lambda\}$ is a linear basis of Λ.

Proof of Proposition. $\left(E(t)H(-t)=1.\right)$

(Involution Principle)

$E(t)\cdot H(-t) =$

$$=\Bigg(\sum_{\substack{k\geq 0\\ i_1<\cdots<i_k}} x_{i_1}\cdots x_{i_k}t^k\Bigg)\Bigg(\sum_{\substack{\ell\geq 0\\ j_1\geq\cdots\geq j_\ell}} x_{j_1}\cdots x_{j_\ell}(-t)^\ell\Bigg)$$

sign reversing involution τ on pairs $(i_1<\cdots<i_k,\ j_1\geq\cdots\geq j_\ell)$

$$\tau:\big((i_1,\dots,i_k),(j_1,\dots,j_\ell)\big)\longmapsto$$

$$\begin{cases}(i_1\dots i_k\, j_1),(j_2,\dots j_\ell) & \text{if } i_k<j_1 \text{ (or } k=0)\\ (i_1,\dots,i_{k-1}),\ (i_k,j_1,\dots,j_\ell) & \text{if } i_k\geq j_1 \text{ (or } \ell=0)\end{cases}$$

τ defined on all pairs, except $(\emptyset,\emptyset)$.

$\tau^2 = \mathrm{id}.$

The involution τ cancels all terms in $E(t)\cdot H(-t)$, except 1 $\leftarrow$ corresponds to $(\emptyset, \emptyset)$. $\square$

2nd proof.

$$E(t) = \sum_{k\geq 0} e_k t^k = \prod_{i=1}^{\infty} (1 + x_i t)$$

$$H(-t) = \sum_{k\geq 0} h_k t^k = \prod_{i=1}^{\infty} (1 - x_i t + (x_i t)^2 - \dots)$$

$$= \prod_{i=1}^{\infty} \frac{1}{1 + x_i t}.$$

Now it is clear that $E(t)H(-t) = 1$

So we proved the h-version of Fund. Thm. of SFs:

$$\Lambda = \mathbb{Z}[h_1, h_2, \dots].$$

Involution ω.

$$\Lambda = \mathbb{Z}[e_1, e_2, \ldots] = \mathbb{Z}[h_1, h_2, \ldots]$$

So we have the ring automorphism $\omega : \Lambda \to \Lambda$ given by

$$\omega : e_k \longmapsto h_k \quad \text{for } k = 1, 2, \ldots$$

(Thus $e_\lambda \mapsto h_\lambda$ $\forall$ partition λ.)

Corollary. (of the symmetry of $E(t)H(-t) = 1$)

ω is an involution.

$$\omega : h_\lambda \longmapsto e_\lambda \quad \forall \lambda.$$

Remark Boson - Fermion corresp.

h_K (all exp. $\in \{0, 1, 2, \ldots\}$) $\quad$ e_K (all exponents $\in \{0, 1\}$)

How about the p-version?

<u>Theorem.</u> $\Lambda_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, \ldots]$.

Equivalently, $\{p_\lambda\}$ is a $\mathbb{Q}$-linear basis of $\Lambda_{\mathbb{Q}}$.

Need to express e_k's (or h_k's) in terms of p_ℓ's.

Let's use generating functions:

$$P(t) := \sum_{k \geq 1} p_k t^{k-1}$$

$$= \sum_{k \geq 1, i \geq 1} x_i^k t^{k-1} = \sum_{i \geq 1} \frac{x_i}{1 - x_i t}$$

..

$$(\log f)' = \frac{f'}{f}$$

$$P(t) = \sum_{i \geq 1} \frac{d}{dt}\left(\log\left(\frac{1}{1-x_i t}\right)\right)$$

$$= \frac{d}{dt}\left(\log\left(\prod_{i=1}^{\infty} \frac{1}{1-x_i t}\right)\right)$$

$$= \frac{d}{dt}\left(\log H(t)\right) = \frac{H'(t)}{H(t)}.$$

Similarly, $P(-t) = \dfrac{E'(t)}{E(t)}$.

We obtain

$$\boxed{\begin{aligned} H'(t) &= P(t)\, H(t) \\ E'(t) &= P(-t)\, E(t) \end{aligned}}$$

or equivalently

Proposition. (Newton's formulas)

$$n \cdot h_n = \sum_{r=1}^{n} p_r h_{n-r}$$

$$n \cdot e_n = \sum_{r=1}^{n} (-1)^{r-1} p_r e_{n-r}$$

These formulas allow to express h_n & e_n as polynomials in p_k's (with rational coeffs), by induction on n.

$$\Longrightarrow \quad \Lambda_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, \dots]$$

More on the involution ω...

$$e_\lambda \xleftrightarrow{\omega} h_\lambda$$

follows from Newton's formulas

$$p_n \xleftrightarrow{\omega} (-1)^{n-1} p_n$$

$$p_\lambda \xleftrightarrow{\omega} (-1)^{|\lambda|-\ell(\lambda)} p_\lambda,$$

where $|\lambda| = \lambda_1 + \dots + \lambda_\ell$

$\ell(\lambda) = \#$ parts in λ.

$$m_\lambda \xleftrightarrow{\omega} ?$$

<u>Definition</u>. <u>Forgotten symmetric functions</u> $f_\lambda := \omega(m_\lambda)$.

$E(t) \cdot H(-t) = 1 \Rightarrow e_n$ can be expressed in terms of h_k's and vise versa.

How to write this expression explicitly?

Let $H = (h_{i-j})_{0 \le i,j \le n}$

and $E = ((-1)^{i-j} e_{i-j})_{0 \le i,j \le n}$

$((n+1) \times (n+1)$ matrices$)$

Convention: $h_0 = e_0 = 1$, $h_r = e_r = 0$, for $r<0$.

$$H = \begin{bmatrix} 1 & & & 0 \\ h_1 & 1 & & \\ h_2 & h_1 & 1 & \\ h_3 & h_2 & h_1 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & & & 0 \\ -e_1 & 1 & & \\ e_2 & -e_1 & 1 & \\ -e_3 & e_2 & -e_1 & 1 \end{bmatrix}$$

lower-triangular matrices, det $= 1$

$E(t) \cdot H(-t) = 1 \iff$ matrices H & E are inverses of each other.

$$E = H^{-1}$$

So the matrix entries of E are the cofactors of the matrix H.

In particular,

$e_n = (-1)^{n-1}$ (lower left entry of E) equals

<u>Proposition</u>

$$e_n = \begin{vmatrix} h_1 & 1 & 0 & 0 & \cdots & 0 \\ h_2 & h_1 & 1 & 0 & \cdots & 0 \\ h_3 & h_2 & h_1 & 1 & \cdots & 0 \\ \cdots & & & & & \cdots \\ h_n & h_{n-1} & h_{n-2} & \cdots & & h_1 \end{vmatrix}$$

$n \times n$ matrix

$$e_1 = \begin{vmatrix} h_1 \end{vmatrix}$$

$$e_2 = \begin{vmatrix} h_1 & 1 \\ h_2 & h_1 \end{vmatrix} = h_1^2 - h_2$$

$$e_3 = \begin{vmatrix} h_1 & 1 & 0 \\ h_2 & h_1 & 1 \\ h_3 & h_2 & h_1 \end{vmatrix} = h_1^3 - 2h_2h_1 + h_3$$

etc.

Applying ω,

$$h_n = \begin{vmatrix} e_1 & 1 & 0 & \cdots & 0 \\ e_2 & e_1 & 1 & \cdots & 0 \\ \cdots & & & & \\ e_n & e_{n-1} & e_{n-2} & \cdots & e_1 \end{vmatrix}.$$

How about explicitly expressing h_n & e_n in terms of p_k's ?

<u>Proposition</u>:

$$h_n = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda ,$$

$$\Big\updownarrow \omega$$

$$e_n = \sum_{\lambda \vdash n} (-1)^{|\lambda| - \ell(\lambda)} z_\lambda p_\lambda ,$$

where $z_\lambda := \prod_{i \geq 1} i^{m_i} m_i!$

and m_i = # parts of λ which are equal to i.

Proof $\quad P(t) = \left(\log H(t)\right)'.$

$$\sum_{r\geq 1} \frac{p_r t^r}{r} = \log H(t)$$

$$H(t) = \exp\left(\sum_{r\geq 1} \frac{p_r t^r}{r}\right)$$

$$= \prod_{r\geq 1} \exp\left(\frac{p_r t^r}{r}\right)$$

$$= \prod_{r\geq 1} \sum_{m_r=0}^{\infty} \frac{(p_r t^r)^{m_r}}{r^{m_r}\, m_r!}$$

$$= \sum_{\substack{(m_1, m_2, \ldots) \\ m_r \geq 0 \ \forall r}} \prod_{r\geq 1} \frac{(p_r t^r)^{m_r}}{r^{m_r}\, m_r!}$$

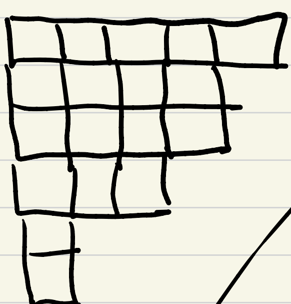
all m_r's but finitely many are equal to 0

Such sequences $(m_1, m_2, \ldots)$ can be identified with partitions λ s.t. λ has m_r parts equal to r $\forall r \geq 1$.

<u>Example</u> $(m_1, m_2, m_3, \ldots) =$

$$= (2, 0, 1, 3, 1, 0, 0, \ldots)$$

$$\rightsquigarrow \lambda = (5, 4, 4, 4, 3, 1, 1)$$

$=$ 

$$|\lambda| = \sum_{r \geq 1} r \cdot m_r$$

So the last expression $\Longleftrightarrow$

$$h_n = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda. \qquad \square$$