last time: $\langle s_\mu \cdot s_\nu, s_\lambda \rangle = \langle s_{\lambda/\mu}, s_\nu \rangle$

The Littlewood-Richardson coefficients

$$c^\lambda_{\mu\nu} := \langle s_\mu \cdot s_\nu, s_\lambda \rangle = \langle s_{\lambda/\mu}, s_\nu \rangle$$

λ, μ, ν partitions s.t. $|\lambda| = |\mu| + |\nu|$.

$$\boxed{s_\mu \cdot s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda}$$

$$\boxed{s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu}$$

There are several rep. theoretical/geometric interpretations of $c^\lambda_{\mu\nu}$:

- In terms of representations of sym. groups: $\mu \vdash m$, $\nu \vdash n$

V_μ, V_ν irreducible reps. of S_m & S_n (Specht modules)

$$V_\mu \circ V_\nu := \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} (\underbrace{V_\mu \otimes V_\nu}_{\text{rep. of } S_m \times S_n})$$

this is not the tensor prod.

rep. of S_{m+n}

$$V_\mu \circ V_\nu = \bigoplus_{\lambda \vdash (m+n)} c^\lambda_{\mu\nu} V_\lambda$$

Here λ, μ, ν are arbitrary partitions s.t. $\mu \vdash m$, $\nu \vdash n$, $\lambda \vdash m+n$

- In terms of rep. of GL_n.

Fix n. $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1 \ldots \mu_n)$, $\nu = (\nu_1, \ldots, \nu_n)$ "partitions" with at most n parts.

$\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$, $\lambda_i \in \mathbb{Z}$ (parts λ_i can be negative)

same for μ & ν.

$V(\lambda)$, $V(\mu)$, $V(\nu)$ irreducible representations of $GL(n)$ (highest weight modules)

$$V(\mu) \otimes V(\nu) = \bigoplus_{\lambda} c^{\lambda}_{\mu\nu} V(\lambda)$$

this is the tensor prod. viewed as a represent. of GL_n

the sum over "partitions" $\lambda = (\lambda_1, \ldots, \lambda_n)$ (n is fixed) s.t. $|\lambda| = |\mu| + |\nu|$

<u>Remark</u>. These $c^{\lambda}_{\mu\nu}$'s are <u>not</u> more general than the LR-coeffs coming from representations of S_n.

- If μ & ν are usual partitions (i.e. they have all nonnegative parts) then λ should be a usual partition (otherwise $c^{\lambda}_{\mu\nu} = 0$).
- $c^{\lambda}_{\mu\nu} = c^{\tilde{\lambda}}_{\tilde{\mu}\tilde{\nu}}$ if

$$\tilde{\mu} = \mu + (a, \ldots, a)$$
$$\tilde{\nu} = \nu + (b, \ldots, b)$$
$$\tilde{\lambda} = \lambda + (a+b, \ldots, a+b)$$

for any $a, b \in \mathbb{Z}$.

$$V(\lambda + (r, \ldots, r)) = V(\lambda) \otimes V(r, \ldots, r)$$

$V(r, \ldots, r)$ is 1-dim representation of GL_n given by

$$\underset{GL_n}{A \in} \longmapsto (\det(A))^r \quad \text{for } r \in \mathbb{Z}.$$

- <u>In terms of Schubert varieties in the Grassmannian.</u>

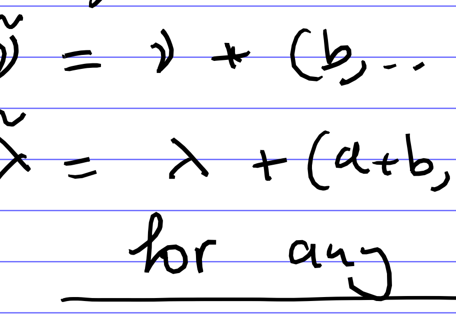
Fix n, k st $n \geq k \geq 0$.

$Gr(k, n)$ - the Grassmannian of k-dim subspaces in $\mathbb{C}^n$.

X_λ - Schubert varieties in $Gr(k,n)$

$\lambda \subseteq k \times (n-k)$.

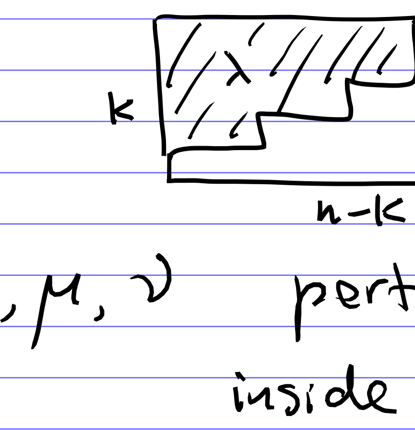
$\lambda = (\lambda_1, \ldots, \lambda_k)$ $\quad \lambda_1 \leq n-k$



λ, μ, ν partitions that fit inside the rectangle $k \times (n-k)$

$\lambda^\vee$ - the complementary partition to λ in the rectangle.

$$\lambda^\vee = (n-k-\lambda_k, n-k-\lambda_{k-1}, \ldots, n-k-\lambda_1)$$
$$= \text{skew diagram } (k \times (n-k)) / \lambda \text{ rotated by } 180°.$$



For $\lambda, \mu, \nu \subseteq k \times (n-k)$

$c_{\lambda\mu\nu}$:= the intersection number of Schubert varieties X_λ, X_μ, X_ν

These numbers $c_{\lambda\mu\nu}$ are related to the LR-coeffs. as

$$c_{\lambda\mu\nu} = c^{\lambda^\vee}_{\mu\nu}$$

These different rep. theoretical / geometric interpretations of the LR coeffs. imply the following properties.

Theorem.

- Nonnegativity: $c^{\lambda}_{\mu\nu}$'s are nonnegative integers.
- Commutativity ($s_\mu \cdot s_\nu = s_\nu \cdot s_\mu$):
  $$c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}$$
- Involution ω:
  $$c^{\lambda}_{\mu\nu} = c^{\lambda'}_{\mu'\nu'}$$ where (λ', μ', ν' are the conjugate partitions)
- S_3-symmetry: For fixed $n \geq k \geq 0$, and $\lambda, \mu, \nu \subseteq k \times (n-k)$ $c_{\lambda\mu\nu}$ is symmetric with respect to any permutation of λ, μ, ν. Thus
  $$\begin{array}{ccccc} c^{\lambda^\vee}_{\mu\nu} & = & c^{\mu^\vee}_{\lambda\nu} & = & c^{\nu^\vee}_{\lambda\mu} \\ \| & & \| & & \| \\ c^{\lambda^\vee}_{\nu\mu} & = & c^{\mu^\vee}_{\nu\lambda} & = & c^{\nu^\vee}_{\mu\lambda} \end{array}$$

So the LR-coeffs. satisfy many different kinds of symmetries. Some of these symmetries are clear from one point of view, but mysterious from another point of view.

For example:

$$s_\mu s_\nu = \sum c^{\lambda}_{\mu\nu} s_\lambda \Rightarrow c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}.$$

$$s_{\lambda/\mu} = \sum c^{\lambda}_{\mu\nu} s_\nu \overset{?}{\not\Rightarrow}$$

Why should the coefficient of s_ν is $s_{\lambda/\mu}$ be equal to the coeffs. of s_μ in $s_{\lambda/\nu}$?

One would like to give a combinatorial formula for the LR-coeffs. & explain all these properties combinatorially.

The Littlewood - Richardson Rule

- a combinatorial rule for $c^{\lambda}_{\mu\nu}$
- "explains" their nonnegativity
- there are several variations of the LR rule related to different ways to think about the LR-coeffs.

The classical LR-rule

Most closely related to the formula:

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} \, s_{\nu}.$$

We have $\quad s_{\lambda/\mu} = \sum_{\nu} K_{\lambda/\mu,\nu} \, m_{\nu}$

the Kostka number

$$K_{\lambda/\mu,\nu} := \#\left\{ \begin{array}{l} \text{SSYT's of} \\ \text{shape } \lambda/\mu \text{ and} \\ \text{weight } \nu \end{array} \right\}$$

Hopefully, $c^{\lambda}_{\mu\nu}$ = the number of some SSYT's of shape λ/μ & weight ν.

Definitions

- A word, i.e., a sequence of positive integers, $w_1 w_2 \dots w_N$ is called a <u>lattice word</u> if in any initial subword $w_1 w_2 \dots w_i$ $\#1's \geq \#2's \geq \#3's \geq \dots$ (Lattice words are also called <u>Yamanouchi</u> <u>words</u>)

Examples: 1 1 2 1 3 2 3

a lattice word ↗

1 1 2 1 3 3 2

<u>not</u> a lattice word ↗

- A <u>Littlewood-Richardson</u> <u>tableau</u> T of shape λ/μ and weight ν is a semistandard Young tableau such that the word obtained by reading the entries of T by rows right-to-left top-to-bottom (reverse reading word of T) is a lattice word.

Let $LR(\lambda/\mu, \nu)$ be the set of such Littlewood-Richardson tableaux.

<u>Example</u>

$$\begin{array}{cccccccccc}
 & & & & & & 1 & 1 & 1 & 1 \\
 & & & 1 & 1 & 2 & 2 & 2 & 2 & \\
 & & & 2 & 2 & 3 & 3 & 3 & 3 & \\
1 & 1 & 1 & 3 & 4 & 4 & 4 & & &
\end{array}$$

a LR-tableau of shape $\lambda/\mu = (10,9,9,7)/(6,3,3)$ and weight $\nu = (9,6,5,3)$

reverse reading word:

$\underbrace{1111}, \underbrace{222211}, \underbrace{333322}, \underbrace{4443111}$

Classical LR rule:

Theorem. $\boxed{c^{\lambda}_{\mu\nu} = \#LR(\lambda/\mu, \nu)}$

the number of Littlewood-Richardson tableaux of shape λ/μ and weight ν.

History • First stated by Littlewood & Richardson 1934

• Robinson 1938 gave a proof with some gaps.

• First complete proofs by Thomas 1974 and Schützenberger 1977.

Example: Suppose $\mu = \emptyset$.

Clearly $S_{\lambda/\emptyset} = S_\lambda$

Let's see why # LR-tableaux of straight shape λ

$$= \begin{cases} 1 & \text{if } \nu = \lambda \\ 0 & \text{otherwise} \end{cases}$$

Indeed, for a straight shape λ, there is only one LR-tableau:

$$\begin{array}{c} \le \\ \wedge \;\; \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \cline{1-4} 3 & 3 & 3 & 3 \\ \cline{1-4} 4 & 4 & 4 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array} \end{array}$$

this is the first entry in rev. reading word, so it should be 1

$\Rightarrow$ 1^{st} row is always filled with 1's

$\Rightarrow$ this entry should by 2

$\Rightarrow$ 2^{nd} row is filled with all 2's

etc.

Example $\lambda = (4,3,2)$, $\mu = (2,1)$,
$\nu = (3,2,1)$

LR-tableaux:

$$\begin{array}{cccc} \cdot & \cdot & 1 & 1 \\ \cdot & \square & 2 & \\ \square & \square & & \end{array}$$

1 or 2

1st row is allways filled with 1st in any LR-tabl.

$$\begin{array}{cccc} \cdot & \cdot & 1 & 1 \\ \cdot & 1 & 2 & \\ 2 & 3 & & \end{array} \qquad\qquad \begin{array}{cccc} \cdot & \cdot & 1 & 1 \\ \cdot & 2 & 2 & \\ 1 & 3 & & \end{array}$$

rev. reading word 1 1 2 1 3 2 — rev. reading word 1 1 2 2 3 1

So $c^{(432)}_{(21),(321)} = 2$.

$$S_{21} \cdot S_{321} = \dots + 2 S_{432} + \dots$$

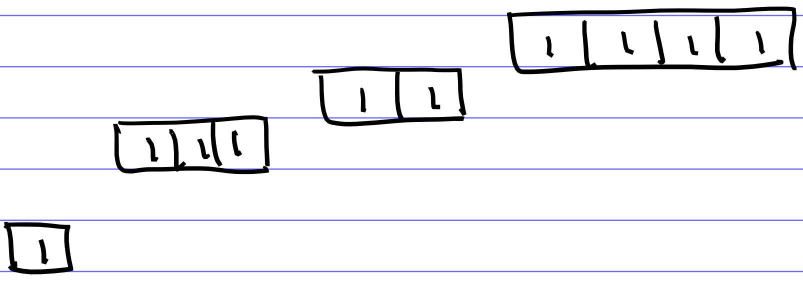
If we fix $n = 3$ and think about LR-rule in $GL(n)$ form, we get

$$c^{432}_{210,\,321} = c^{321}_{210,\,210}$$

$$S_{(2,1)} \cdot S_{(2,1)} = S_{(4,2)} + S_{(4,1,1)} + S_{(3,3)} + \underline{\underline{2}} \cdot S_{(3,2,1)} + S_{(3,1,1,1)} + S_{(2,2,2)} + S_{(2,2,1,1)}$$

<u>Example</u> Assume that $\nu = (k)$

A LR-tableau of weight (k) should be a horizonal k-strip filled with 1's

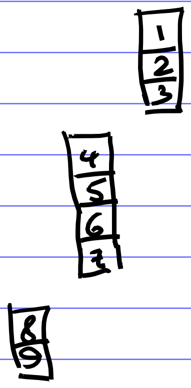


In this case, LR-rule $\Rightarrow$ Pieri's rule:

$$S_k \cdot S_\mu = \sum_{\substack{\lambda \text{ s.t. } \lambda/\mu \text{ is a} \\ \text{horizontal } k\text{-strip}}} S_\lambda$$

h_K

If $\nu = 1^k$, then any LR-tableaux should be a vertical k-strip filled with $1, 2, \dots, k$ from to bottom



So LR-rule implies 2^{nd} Pieri's rule:

$$S_{1^k} \cdot S_\mu = \sum_{\substack{\lambda:\ \lambda/\mu \text{ is a} \\ \text{vertical } k\text{-strip}}} S_\lambda$$

e_K

<u>Remark</u>. In this classical LR-rule, all symmetries that we mentioned (commutativity, inv. ω, S_3-sym.) are pretty non-obvious.

Some variations / generalizations of the LR-rule:

Zelevinsky's pictures

LR-rule is a rule for $\langle s_{\lambda/\mu}, s_\nu \rangle$.

How about a rule for the inner product of any two skew Schur functions

$$\langle s_{\lambda/\mu}, s_{\nu/\gamma} \rangle = ?$$

Of course, this number can be expressed in terms of the LR-coefficients:

$$s_{\lambda/\mu} = \sum_{\varkappa} c^{\lambda}_{\mu\varkappa} s_{\varkappa}$$

$$s_{\nu/\gamma} = \sum_{\varkappa} c^{\nu}_{\gamma\varkappa} s_{\varkappa}$$

So $\langle s_{\lambda/\mu}, s_{\nu/\gamma} \rangle =$

$$= \sum_{\varkappa} c^{\lambda}_{\mu\varkappa} c^{\nu}_{\gamma\varkappa}.$$

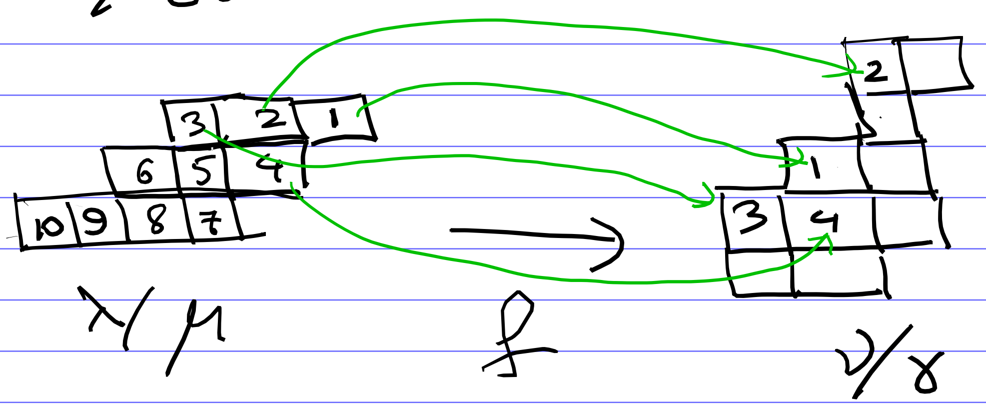
But we would like to give a better rule.

<u>Definition</u> A <u>Zelevinsky picture</u> is a bijection f between boxes of two skew shapes λ/μ and ν/γ:

$$f : \lambda/\mu \longrightarrow \nu/\gamma$$

s.t.

(A) If we fill the shape λ/μ with $1, 2, \ldots, N = |\lambda/\mu|$ by rows right-to-left top-to-bottom (the same order as in the reverse reading), then the images of the labels under the map f form a SYT of shape ν/γ.



(B) The same condition for the inverse map f^{-1}

Clearly, # of maps f satisfying just the condition (A) = # SYT's of shape ν/γ

and # of maps f satisfying (B) = # SYT's of shape λ/μ.

But the conditions (A) & (B) together give some new number.

Theorem (Zelevinsky)

$\langle s_{\lambda/\mu}, s_{\nu/\gamma} \rangle =$
Zelevinsky pictures
$f: \lambda/\mu \longrightarrow \nu/\gamma$.

Let's show that this "picture rule" generalizes the classical LR-rule

Claim If $\gamma = \emptyset$, then Zelevinsky's pictures are in bijection with LR-tableaux of shape λ/μ and weight ν.

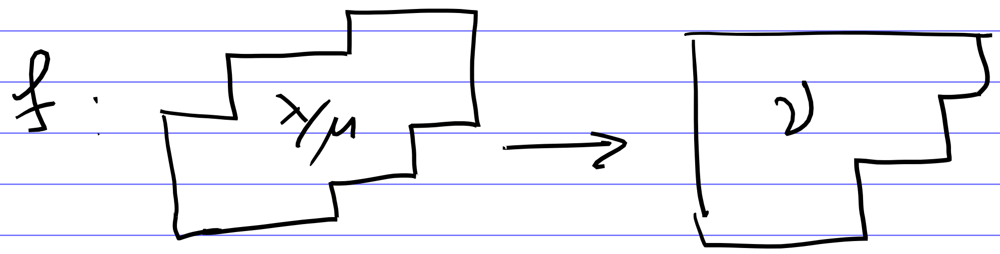


tableau T of shape λ/μ
s.t. a box x of λ/μ is filled with i if $f(x) \in$ i^{th} row of ν.

Check: In this case, the conditions (A) & (B) are equiv. to the definition of LR-tableaux. (Exercise)

Berenstein-Zelevinsky triangles

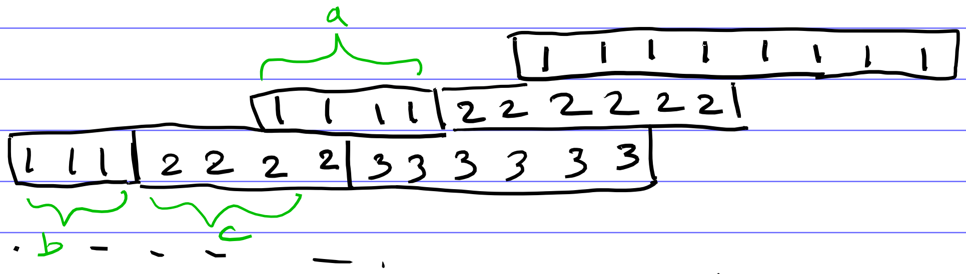
Goal: To reformulate the LR-rule in a more symmetric form (that would explain some symmetries of the LR-coeffs.)

Idea: Recast the definition of LR-tableaux in terms of Gelfand-Tsetlin patterns.

We will use the "$GL(n)$-version" of the LR-coefficients.

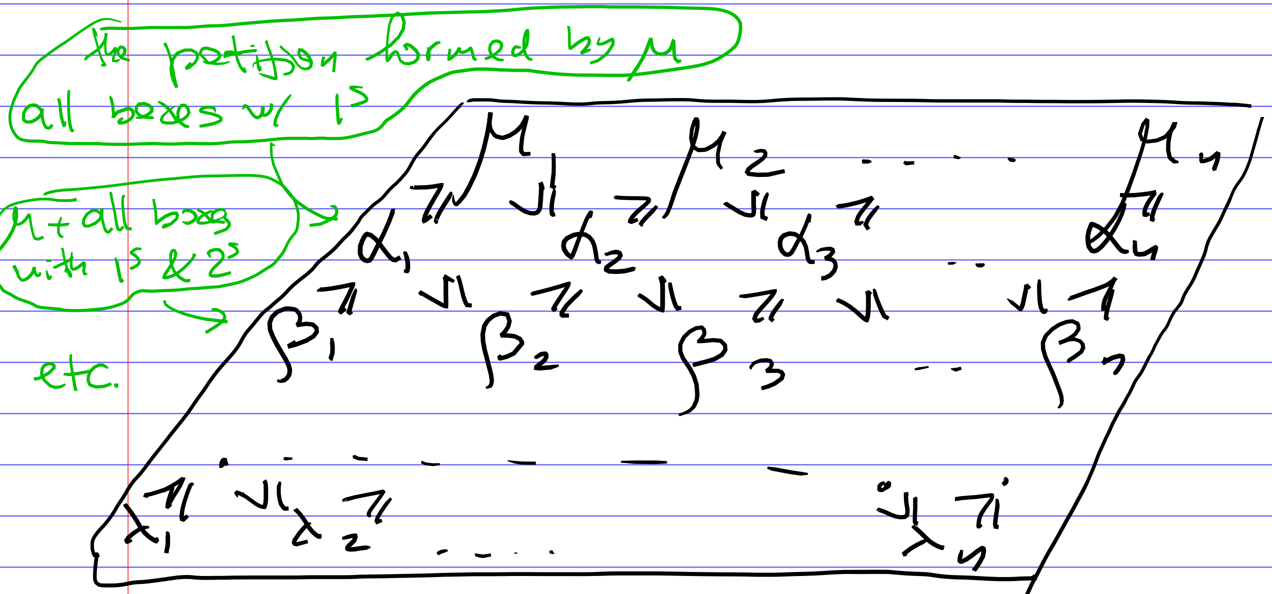
Fix n. λ, μ, ν are partitions with at most n parts:

- λ/μ has at most n rows
- LR-tableaux are filled with numbers $\in \{1, \dots, n\}$.

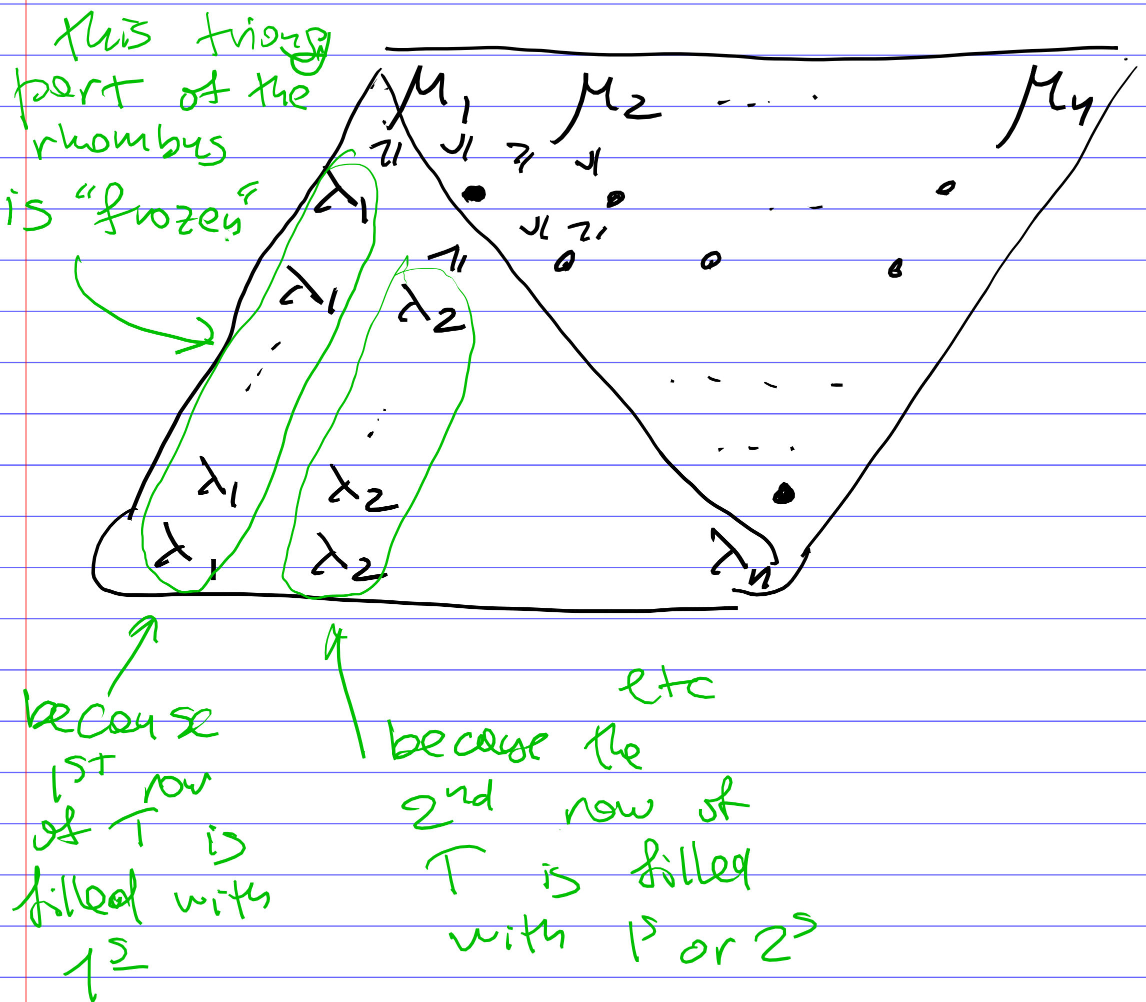


a LR-tableau T of shape λ/μ.

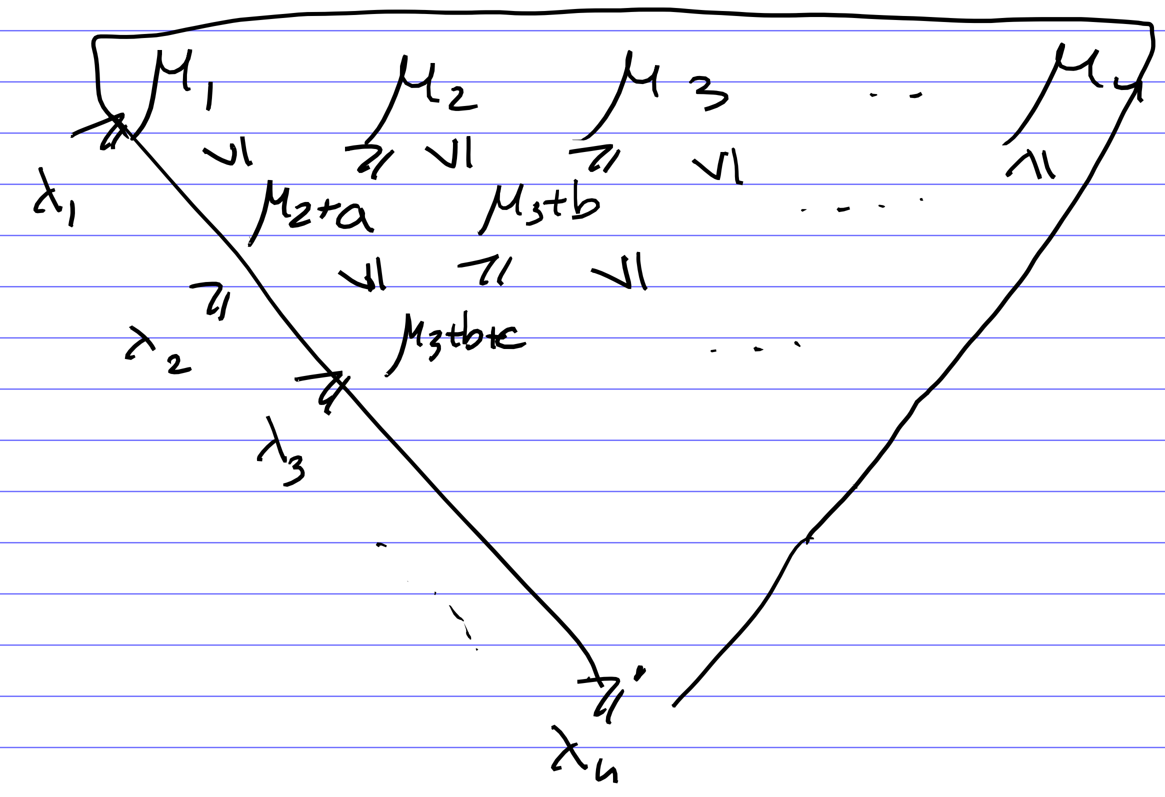
T is a SSYT, so it can be written as a Gelfand-Tsetlin-like pattern of rhombic form:



The LR-conditions (i.e. the lattice word conditions) imply some additional equalities & inequalities for entries of this pattern:



The "interesting" part of the pattern has triangular form:



reverse reading word:

$$\overbrace{\underbrace{1\ldots1}_{\lambda_1-\mu_1}}^{1^{st}\text{ row}}\ \overbrace{\underbrace{2\ldots2}_{\lambda_2-\mu_2-a}\ \underbrace{1\ldots1}_{a}}^{2^{nd}\text{ row}}\ \overbrace{\underbrace{3\ldots3}_{\lambda_3-\mu_3-b-c}\ \underbrace{2\ldots2}_{c}\ \underbrace{1\ldots1}_{b}}^{3^{rd}\text{ row}}\ldots$$

lattice word conditions:

$$\lambda_1-\mu_1 \geq \lambda_2-\mu_2-a$$

$$\lambda_2-\mu_2-a \geq \lambda_3-\mu_3-b-c$$

$$\lambda_1-\mu_1+a \geq \lambda_2-\mu_2-a+c$$

etc.

So

- # LR-tableaux = # triangule GT patterns with some additional linear inequalities (coming from lattice conditions)...

In the next lecture we'll show that all these inequalities can be written explicitly in a very symmetric form...