18.217 Lecture 26 11/02/2020

Last time: We proved the M-N rule for symmetric functions modulo some claims about the inner product on Λ.

Definition. The Hall inner product $\langle \cdot, \cdot \rangle$ on the space of symmetric functions Λ (in ∞ many variables) is defined by

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$$

the Kronecker delta function $\delta_{\lambda\mu} = \begin{cases} 1, & \lambda = \mu \\ 0, & \lambda \neq \mu \end{cases}$

i.e. the bases $\{h_\lambda\}$ and $\{m_\lambda\}$ of Λ are dual w.r.t. $\langle \cdot, \cdot \rangle$.

Lemma. $\displaystyle\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} =$

this is called the Cauchy product

$$= \sum_{\lambda \text{ partition}} s_\lambda(x)\, s_\lambda(y)$$

(the Cauchy form. we proved it using the RSK)

the sums over all partitions λ, including $\varnothing$,

$$= \sum_{\lambda \text{ part.}} m_\lambda(x)\, h_\lambda(y)$$

$$= \sum_{\lambda \text{ part.}} \frac{1}{z_\lambda} p_\lambda(x)\, p_\lambda(y),$$

(where $z_\lambda = \prod_{i \geq 1} (i^{m_i} m_i!)$,

$m_i = \#\{\text{parts equal to } i \text{ in } \lambda\}$).

<u>Proof.</u> Lets calulate the coeff. of a monomial $x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ (any seq. of integers) in the Cauchy product $\prod_{i,j} \frac{1}{1-x_i y_j} =$

$$= \prod_{i,j} \left(1 + (x_i y_j) + (x_i y_j)^2 + \cdots\right).$$

It is easy to see that the coff. of x^{α} equals $h_{\alpha_1}(y) h_{\alpha_2}(y) \cdots$

$$\Rightarrow \prod_{i,j} \frac{1}{1-x_i y_j} = \sum m_\lambda(x) h_\lambda(y).$$

the sum over all perditions λ.

$$\sum_\lambda \frac{1}{z_\lambda} p_\lambda(x) p_\lambda(y) =$$

$$= \sum_{m_1, m_2, \ldots \geq 0} \prod_{i \geq 0} \frac{1}{i^{m_i} m_i!} (x_1^i + x_2^i + \cdots)^{m_i} \cdot (y_1^i + y_2^i + \cdots)^{m_i}$$

$\frac{1}{z_\lambda}$

$$= \prod_{i \geq 1} \left(\sum_{m_i \geq 0} \frac{1}{m_i!} \left(p_i(x) \cdot p_i(y) \cdot \frac{1}{i} \right)^{m_i} \right)$$

$$= \prod_{i \geq 1} e^{p_i(x) \cdot p_i(y) \cdot \frac{1}{i}} = e^{\sum_{i \geq 1} p_i(x) \cdot p_i(y) \frac{1}{i}}$$

$$= \prod_{a,b \geq 1} e^{\sum_{i \geq 1} \frac{1}{i} x_a^i x_b^i}$$

$$-\log(1-z) = \sum_{i \geq 1} \frac{z^i}{i}$$

$$= \prod_{a,b \geq 1} e^{\log(1 - x_a y_b)^{-1}}$$

$$= \prod_{a,b} (1 - x_a y_b)^{-1}, \text{ as needed. } \square$$

Lemma Two linear bases $\{u_\lambda\}$ and $\{S_\lambda\}$ of $\Lambda_{\mathbb{R}} := \Lambda \otimes \mathbb{R}$ are dual to each other w.r.t. the Hall product $\langle \cdot , \cdot \rangle$ iff

$$(*) \quad \sum_\lambda{}' u_\lambda(x) \cdot S_\lambda(y) = \prod_{i,j \geq 1} \frac{1}{1-x_i y_j}.$$

Proof Let us express u_λ and S_μ in terms of bases $\{h_\nu\}$ and $\{m_\gamma\}$, respectively:

$$u_\lambda = \sum_\nu a_{\lambda\nu} h_\nu$$

$$S_\mu = \sum_\gamma b_{\mu\gamma} m_\gamma$$

where $a_{\lambda\nu}$ and $b_{\mu\nu}$ are some coeffs. $\in \mathbb{R}$

Then

$$\langle u_\lambda , S_\mu \rangle = \delta_{\lambda\mu}$$

$$\Leftrightarrow \langle \sum_\nu a_{\lambda\nu} h_\nu , \sum_\gamma b_{\mu\gamma} m_\gamma \rangle = \delta_{\lambda\mu}$$

$$\Leftrightarrow \sum_\nu a_{\lambda\nu} \cdot b_{\mu\nu} = \delta_{\lambda\mu}$$

$$\Leftrightarrow A B^T = I, \qquad A = (a_{\lambda\nu}), \quad B = (b_{\mu\nu}) \text{ matrices.}$$

On the other hand,

$$(*) \Leftrightarrow \sum_\lambda u_\lambda(x) S_\lambda(y) = \sum_\lambda h_\lambda(x) m_\lambda(y)$$

$$\| $$

$$\sum_\lambda \left(\sum_\nu a_{\lambda\nu} h_\nu(x)\right)\left(\sum_\gamma b_{\lambda\nu} m_\gamma(y)\right)$$

$$\Leftrightarrow \sum_\nu a_{\lambda\nu} b_{\lambda\gamma} = \delta_{\nu\gamma}$$

$$\Leftrightarrow A^T B = I$$

Now, it is clear that

$$A B^T = I \Leftrightarrow B = (A^{-1})^T$$

$$\Leftrightarrow A^T B = I. \qquad \square$$

We obtain.

<u>Theorem</u>. The Hall inner product $\langle \cdot , \cdot \rangle$ satisfies:

- $\boxed{\langle s_\lambda , s_\mu \rangle = \delta_{\lambda\mu}}$

  i.e. the Schur functions s_λ for an orthonormal basis of Λ

- $\boxed{\langle p_\lambda , p_\mu \rangle = z_\lambda \delta_{\lambda\mu}}$

  i.e. p_λ's form an orthogonal basis, and $z_\lambda^{-1/2} p_\lambda$'s form an orthonormal basis of $\Lambda_{\mathbb{R}}$.

- $\boxed{\langle h_\lambda , m_\mu \rangle = \delta_{\lambda\mu}}$

  i.e. $\{h_\lambda\}$ and $\{m_\mu\}$ form a pair of dual bases of Λ

  (This is the def. of $\langle \cdot , \cdot \rangle$.)

Theorem. For any $f \in \Lambda$,

$$(**) \quad \boxed{\langle s_\lambda, f \cdot s_\mu \rangle = \langle s_{\lambda/\mu}, f \rangle}.$$

In particular, for $f = s_\nu$, we have

$$(***) \quad \boxed{\langle s_\lambda, s_\mu \cdot s_\nu \rangle = \langle s_{\lambda/\mu}, s_\nu \rangle}$$

(Actually, $(**) \Leftrightarrow (***)$ because $\{s_\nu\}$ is a basis of Λ.)

Example 1. Take $f = s_\square = h_1 = x_1 + x_2 + \dots$.

Let $h_1 \cdot s_\mu = \sum c_{\lambda\mu} s_\lambda$

$$(**) \Leftrightarrow \quad c_{\lambda\mu} = \langle h_1 \cdot s_\mu, s_\lambda \rangle$$
$$= \langle s_{\lambda/\mu}, s_\square \rangle$$
$$= \begin{cases} 1 & \text{if } \lambda \gtrdot \mu \\ 0 & \text{otherwise} \end{cases}$$

covering relation in Young's lattice $\mathbb{Y}$.

So $(**) \Leftrightarrow$ $\boxed{h_1 \cdot s_\mu = \sum_{\lambda \gtrdot \mu} s_\lambda}$

Example 2. Take $\mu = \square$

$(***) \Leftrightarrow \quad \langle s_\lambda, s_\square s_\nu \rangle =$

$$= \langle s_{\lambda/(1)}, s_\nu \rangle$$

Also $\langle s_\lambda, s_\square s_\nu \rangle = \begin{cases} 1 & \nu \lessdot \lambda \\ 0 & \text{otherwise} \end{cases}$

by the previous example

So we get

$$s_{\lambda/(1)} = \sum_{\nu \lessdot \lambda} s_\nu$$

i.e. ν is obtained from λ by removing a corner box

$$s_{(2,1)/(1)} = s_{(2)} + s_{(1,1)} \qquad (\lambda = (2,1))$$

$$s_{(3,2)/(1)} = s_{(2,2)} + s_{(3,1)} \qquad (\lambda = (3,2))$$

Expanding these Schur function in monomials we obtain

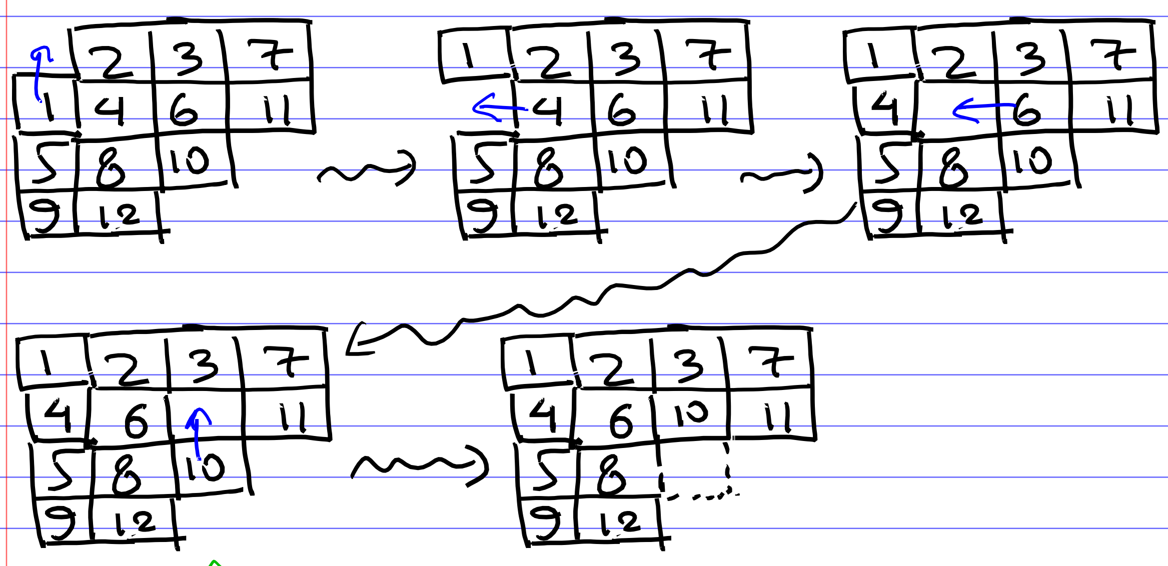
Corollary $\forall$ partitions λ an β, s.t. $|\lambda| = |\beta| + 1$, we have

$$\#\left\{\begin{array}{l}\text{SSYT's of} \\ \text{skew shape } \lambda/(1) \\ \text{and weight } \beta\end{array}\right\} = \sum_{\nu \text{ s.t. } \nu \lessdot \lambda} \#\left\{\begin{array}{l}\text{SSYT's} \\ \text{of shape } \nu \\ \text{and weight } \beta\end{array}\right\}.$$

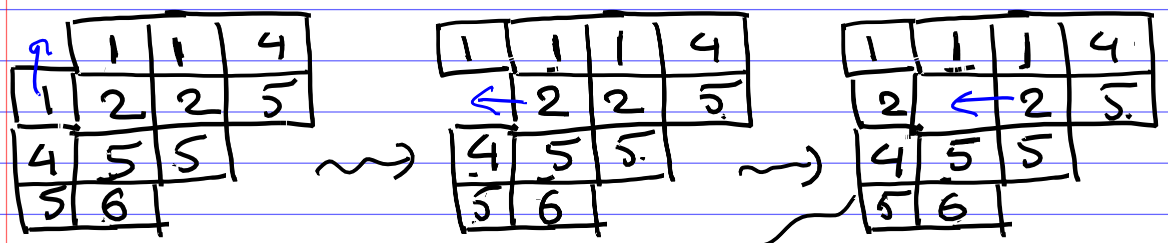
A bijective proof?

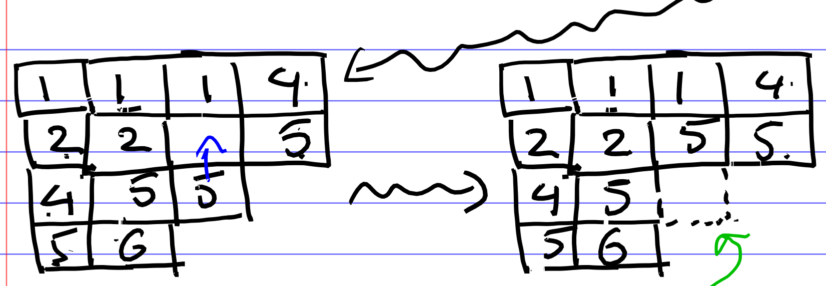
Jeu de taquin (the game of 15)

Example



This is an example for SYT's.
Same game works for SSYT's





this is called jeu de taquin "evacuation"

This game is reversable (if the know the position of the last box that was emptied)

More generally for $\mu = \underbrace{\square\square\square\square}_{K}$

(**) gives

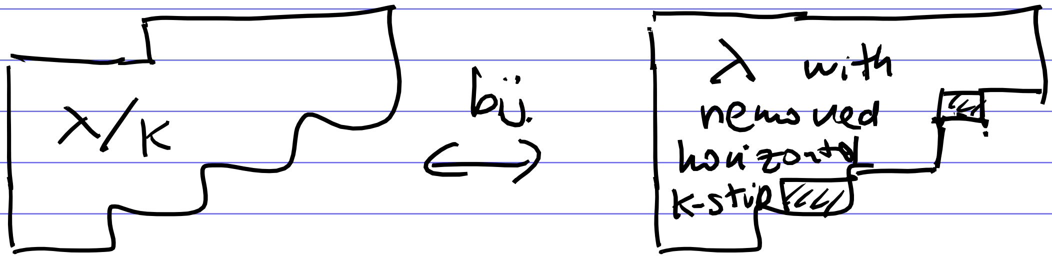
Corollary

$$s_{\lambda/(K)} = \sum_{\substack{\nu \text{ obtained from} \\ \lambda \text{ by removing a} \\ \text{horizontal } K\text{-strip.}}} s_\nu$$

Combinatorially we get

$$\#\left\{\begin{array}{l}\text{SSYT's of} \\ \text{shape } \lambda/K \text{ \&} \\ \text{weight } \beta\end{array}\right\}$$

$$= \sum_{\substack{\nu \text{ obtained from} \\ \lambda \text{ by removing} \\ \text{a horizontal} \\ K\text{-strip}}} \#\left\{\begin{array}{l}\text{SSYT's} \\ \text{of shape} \\ \nu \text{ \& weight} \\ \beta\end{array}\right\}$$



SSYT's SSYT's

A more general claim:

Proposition. For any skew Young diagram λ/μ and any positive int. k.

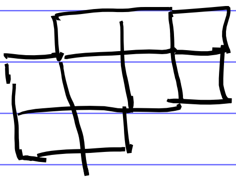
$$\sum_{\substack{\tilde{\mu}:\ \mu \subseteq \tilde{\mu} \subseteq \lambda \\ \tilde{\mu}/\mu \text{ is a horizontal} \\ k\text{-strip}}} s_{\lambda/\tilde{\mu}} =$$

$$= \sum_{\substack{\tilde{\lambda}:\ \mu \subseteq \tilde{\lambda} \subseteq \lambda \\ \lambda/\tilde{\lambda} \text{ is a horizontal} \\ k\text{-strip}}} s_{\tilde{\lambda}/\mu}$$

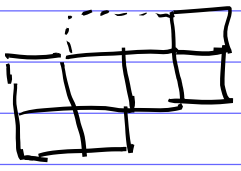
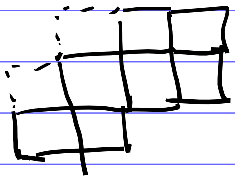
Claim. Actually this Prop. implies the formula

$$\langle s_\mu \cdot f,\ s_\lambda \rangle = \langle f, s_{\lambda/\mu} \rangle.$$

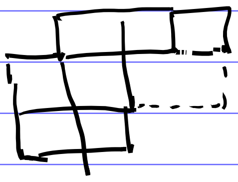
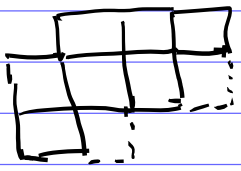
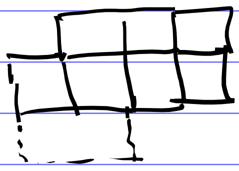
Exercise Show this.

Example $\lambda/\mu =$ 

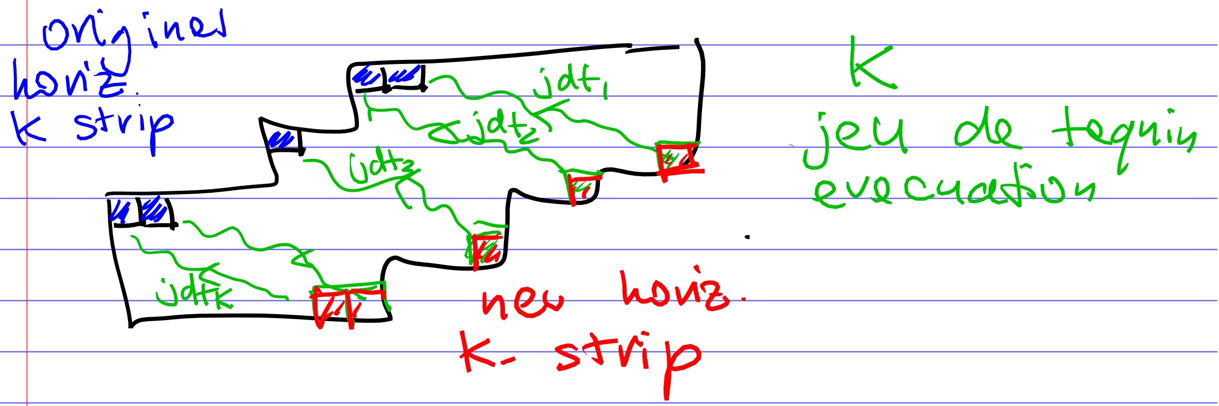
$k = 2$

 +  =

Here we mean that we take sums of skew Schur functions

=  +  + 

This proposition can also be proved combinatorially by constructing bijections between SSYT's using jeu de taquin.



We need to do a jdt- evacuation for each blue box starting from the rightmost blue box, then next rightmost, etc.

<u>Lemma</u> The k evacuation routes don't cross each other.

Thus the red boxes (the boxes that were emptied last in each evacuation) are also arranged from right-to-left and form a horizontal k strip.

Another way to prove Prop. is to use the "up operators" acting on Λ by

$$\boxed{U_k : s_\mu \mapsto \sum_{\substack{\lambda \supseteq \mu \\ \lambda/\mu \text{ horizontal} \\ k \text{ strip}}} s_\lambda}$$

Pieri rule: $U_k(s_\mu) = h_k \cdot s_\mu$.

<u>Lemma</u> The operators U_k, $k \geq 1$ commute with each other.

<u>Proof of Proposition</u>

$$\sum_{\substack{\tilde{\mu}: \\ \tilde{\mu}/\mu \text{ k-strip}}} s_{\lambda/\tilde{\mu}} \overset{?}{=} \sum_{\substack{\tilde{\lambda}: \\ \lambda/\tilde{\lambda} \text{ k-strip}}} s_{\tilde{\lambda}/\mu}$$

The coeffs of $x_1^{\beta_1} x_2^{\beta_2} \cdots x_r^{\beta_r}$ are

<u>in LHS:</u> The coeff of s_λ in

$$U_{\beta_r} \cdots U_{\beta_2} U_{\beta_1} U_k (s_\mu)$$

<u>in RHS:</u> The coeff of s_λ in

$$U_k U_{\beta_r} \cdots U_{\beta_2} U_{\beta_1} (s_\mu).$$

Clearly, these two coeffs are equal because U_ℓ's commute with each other. $\square$

Moreover, we constructed explicit bijections between SSYT's showing that

$$U_k U_\ell = U_\ell U_k$$

called the <u>Bender-Knuth involutions</u>

Recall:

$$\begin{array}{ccccccccc} & & \mu_1 & & \mu_2 & & \mu_3 & \cdots & \\ & \nu_1 & & \nu_2 & & \nu_2 & & \cdots & \\ \lambda_1 & & \lambda_2 & & \lambda_2 & & \lambda_3 & \cdots & \end{array}$$

add hor. k-strip

add hor. ℓ-strip

<u>Bender-Knuth involution</u>

$$\begin{array}{ccccccccc} & & \mu_1 & & \mu_2 & & \mu_3 & \cdots & \\ & \nu_1^* & & \nu_2^* & & \nu_3^* & & \cdots & \\ \lambda_1 & & \lambda_2 & & \lambda_3 & & \lambda_3 & \cdots & \end{array}$$

add hor. ℓ-strip

add hor. k-strip

where ν_i^*'s are given by toggles of ν_i's

$$\nu_i^* = \min(\mu_{i-1}, \lambda_i) + \max(\mu_i, \lambda_{i+1}) - \nu_i$$

(Assuming $\mu_0 = \infty$).

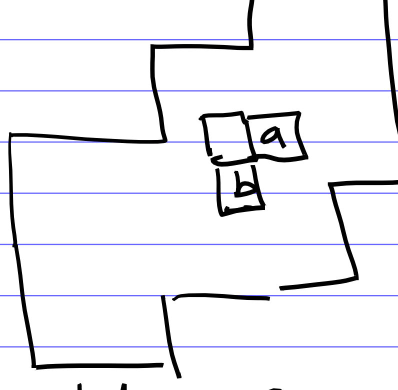
So we obtain 2 bijection proofs of the proposition:

- the bijection obtained by k jeu de taquin evacuations.
- the bijection obtained by the composition of r Bender-Knuth involutions (r = # parts in the weight of SSYT's)

<u>Exercise.</u> Are these 2 bijections equal to each other? What is the relationship between the <u>toggle operations</u> (elementary steps in Bender-Knuth involutions) and the <u>jdt-slides</u> (elementary steps in evaluations)?

<u>Toggles:</u> ? <u>Slides:</u>

$$\begin{array}{ccc} a & & b \\ & e & \\ c & & d \end{array}$$



$e \leftrightarrow e^* = \min(a,c) + \max(b,d) - e.$

slide $\leftarrow$ a if $a < b$, or slide $\uparrow$ b if $b \le a$

Recall (from the last lecture)

Frobenius char. map

$$ch: \left\{\begin{array}{c}\text{class functions} \\ \text{on } S_n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{symm.} \\ \text{functions} \\ \text{of deg. } n\end{array}\right\}$$

(or virtual characters of S_n)

$$ch: \quad \chi_\lambda := \chi_{V_\lambda} \longmapsto s_\lambda$$

irreps of S_n — Schur funct.

$$\chi(w) = \begin{cases} 1 & \text{if type}(w) = \mu \\ 0 & \text{otherwise} \end{cases} \longmapsto \frac{p_\mu}{z_\lambda}$$

(rescaled) power sym. functions

Operations on representations of symm. group

$\updownarrow$

operations on symmetric functions.

Direct sum of reps. $V \oplus W$

$$ch(\chi_{V \oplus W}) = \underbrace{ch(\chi_V) + ch(\chi_W)}_{\text{sum of sym}}$$

But for tensor products

$$ch(\chi_{V \otimes W}) \neq ch(\chi_V) \cdot ch(\chi_W)$$

Tensors products of reps. corresponds to a new operation $f * g$ on symmetric functions called the Kronecker product

The coeffs. $g_{\lambda\mu}^{\nu} \in \mathbb{Z}_{\geq 0}$

$$V_\lambda \otimes V_\mu = \sum g_{\lambda\mu}^{\nu} V_\nu$$

are called the Kronecker coeffs

$$s_\lambda * s_\mu = \sum_\nu g_{\lambda\mu}^{\nu} \, s_\nu$$

$$s_\lambda * s_\mu \neq s_\lambda \cdot s_\mu$$

For example

$$\deg(s_\lambda * s_\mu) = \deg(s_\lambda) = \deg(s_\mu)$$

but $\deg(s_\lambda \cdot s_\mu) = \deg(s_\lambda) + \deg(s_\mu)$.

Induced Representations

V a rep. of S_m

W a rep. of S_n

$V \otimes W$ a rep. of $S_m \times S_n$

$$V \circ W := \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} (V \otimes W)$$

rep. of S_{m+n}

Then $\mathrm{ch}(V \circ W) = \mathrm{ch}(V) \cdot \mathrm{ch}(W)$

Example $V_\square$ trivial rep of S_1

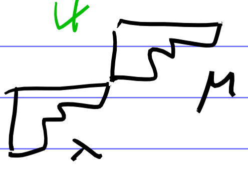
$$\underbrace{V_\square \circ V_\square \circ \cdots \circ V_\square}_{n \text{ times}} \overset{\text{def}}{:=}$$

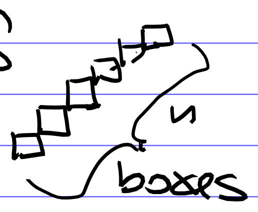
$$= \mathrm{Ind}_{\{id\}}^{S_n} 1 = \begin{pmatrix} \text{the regular} \\ \text{representation of} \\ S_n \end{pmatrix}$$

$$\mathrm{ch}\begin{pmatrix} \text{the regular} \\ \text{rep. of } S_n \end{pmatrix} = (s_\square)^n$$

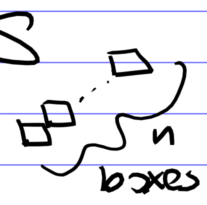
In terms of skew diagrams:

skew Young diagr. which is disjoint union of 2 diagr.

$$S_\lambda \cdot S_\mu = S$$ 

$$\mathrm{ch}\begin{pmatrix}\text{reg. rep.}\\ \text{of } S_n\end{pmatrix} = S$$ 

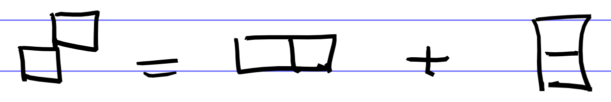
<u>Proposition</u> (<u>2 decompositions of the regular rep.</u>)

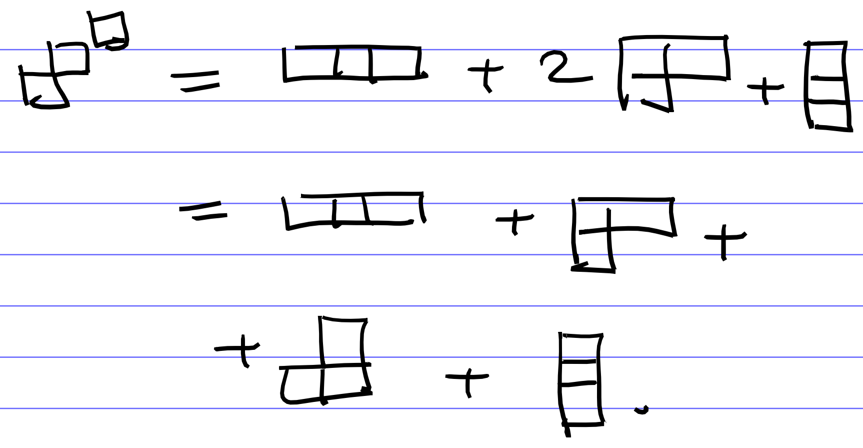
$$S \;=\; \sum_{\lambda \vdash n} \#SYT(\lambda)\cdot S_\lambda$$ 

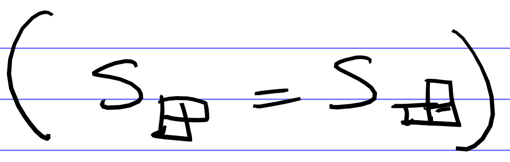
$$= \sum_{\substack{\lambda/\mu \text{ ribbon}\\ \text{with } n \text{ boxes}\\ \text{(considered up to}\\ \text{parallel translation)}}} S_{\lambda/\mu}.$$

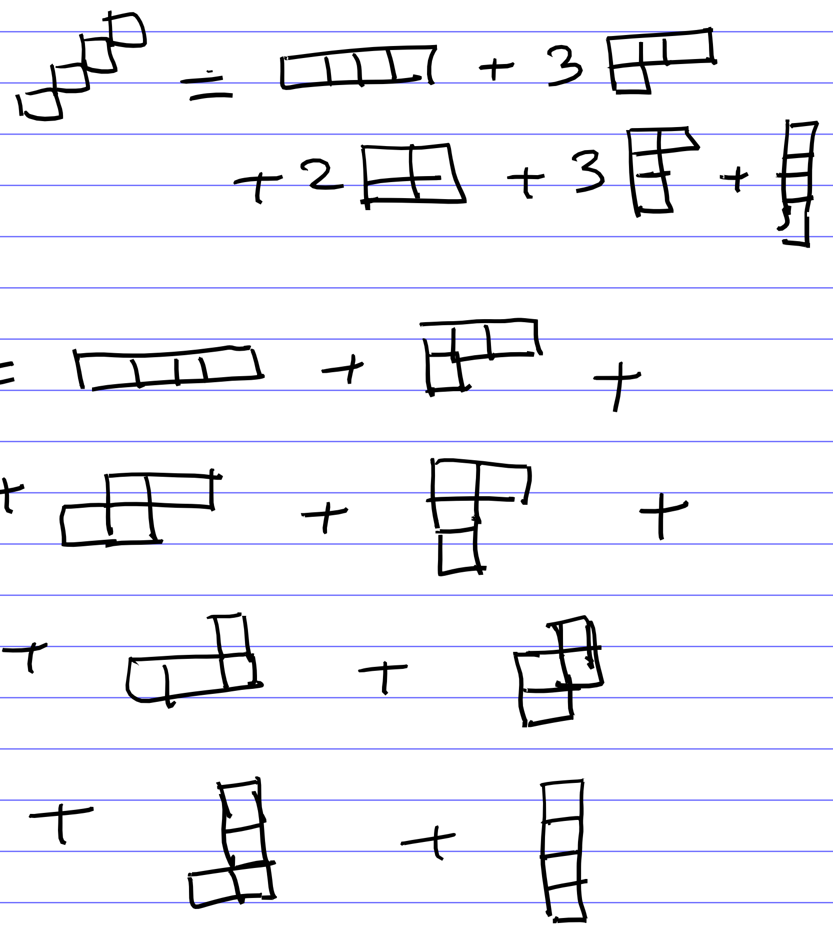
<u>Exercise</u> Give a bijective proof of this claim.

<u>Example</u>

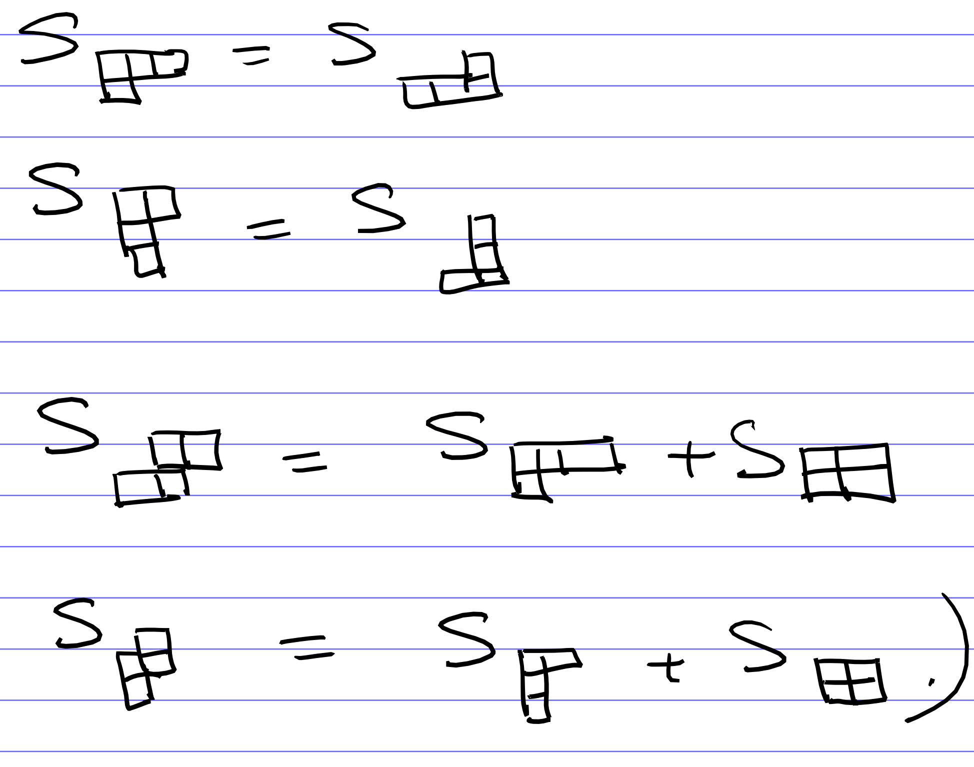
$n=2$ 

$n=3$ 



$n=4$: 

(Some identities: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