Recall: V_λ's, for $\lambda \vdash n$, are the irreducible representations of S_n (also called Specht modules)

Usually, "Specht modules" refer to one particular construction of V_λ's in terms of tabloids ~ Young's symmetrizer.

$V_{\lambda/\mu}$ more general representations of S_n labelled by skew Young diagrams (with n boxes)

$\chi_{\lambda/\mu} = \chi_{V_{\lambda/\mu}}$ is the charater of $V_{\lambda/\mu}$, which is a class function on S_n, i.e. a function $\chi: S_n \to \mathbb{C}$ constant on conjugacy classes.

We'll denote the values of the characters by $\chi_{\lambda/\mu}(w) = \chi_{\lambda/\mu}(\nu)$ where the partition $\nu = \text{type}(w)$ is the cyclic type of permutation $w \in S_n$

$$w = \underbrace{(a_1 \dots a_{\nu_1})}_{\nu_1} \underbrace{(b_1 \dots b_{\nu_2})}_{\nu_2} \dots \underbrace{(\dots)}_{\nu_k}$$

$$\text{type}(w) := \nu = (\nu_1, \dots, \nu_k) \vdash n.$$

$\{\chi_\lambda\}_{\lambda \vdash n}$ is a linear basis in the space of class functions on S_n.

Arbitratry class functions $\chi: S_n \to \mathbb{C}$ are also called virtual characters because they linear (but not nec. positive) combinations of χ_λ's.

Murnaghan-Nakayama Rule:

$$\chi_{\lambda/\mu}(\nu) = \sum_{\substack{RT \text{ ribbon tableau} \\ \text{of shape } \lambda/\mu \\ \text{and type } \nu}} (-1)^{ht(RT)}$$

... back to symmetric functions ...

Frobenius characteristic map
(aka Frobenius character)

$$ch: \left\{ \begin{matrix} \text{class functions} \\ \text{on } S_n \end{matrix} \right\} \to \Lambda^n_{\mathbb{C}}$$

the space of symmetric functions of degree n with complex coeffs.

For a class function $\chi: S_n \to \mathbb{C}$,

$$\boxed{ch(\chi) := \frac{1}{n!} \sum_{w \in S_n} \chi(w) \, p_{type(w)}}$$

where $p_\lambda := p_{\lambda_1} \cdot p_{\lambda_2} \cdots p_{\lambda_k}$ power symmetric functions for $\lambda = type(w)$.

Equivalently,

$$\boxed{ch(\chi) := \sum_{\lambda \vdash n} \chi(\lambda) \, \frac{p_\lambda}{z_\lambda}}$$

where $\frac{1}{z_\lambda} := \frac{1}{n!} \cdot \# \left\{ \begin{matrix} \text{permutations in } S_n \\ \text{with cyclic type } \lambda \end{matrix} \right\}$

$= \text{Prob} \left(\begin{matrix} \text{random (uniformly chosen) permutation} \\ \text{in } S_n \text{ has} \\ \text{cyclic type } \lambda \end{matrix} \right)$

Lemma. $\boxed{z_\lambda = \prod_{i \geq 1} i^{m_i} \, m_i!}$,

where $m_i = \#$ parts in λ of size i.

Proof Fix a permutation $w \in S_n$ with $type(w) = \lambda$.

$\frac{n!}{z_\lambda} \overset{def}{=} \# \left\{ \begin{matrix} \text{permutations in } S_n \\ \text{with cyclic type } \lambda \end{matrix} \right\}$

$=$ the size of the orbit $Orbit(w)$ of w under the action

of S_n on itself by conjugations:

$$u: w \longmapsto u w u^{-1}$$

$$Orib(w) := \{ u w u^{-1} \mid u \in S_n \}$$

From group theory, we know

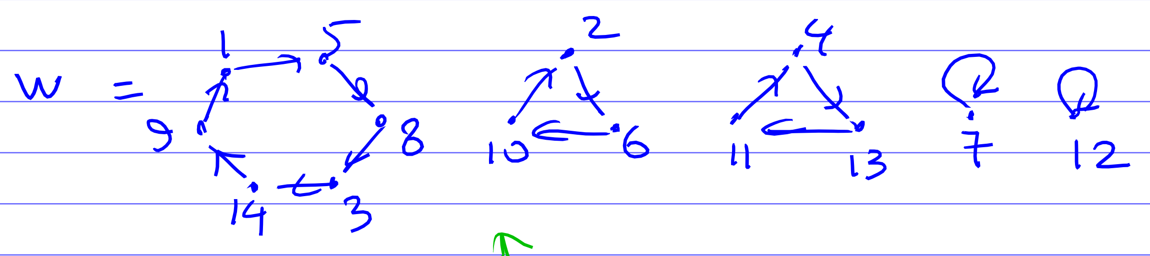
$$|Orbit(w)| = \frac{|S_n|}{|Fix(w)|},$$

where $Fix(w) := \{ u \in S_n \mid u w u^{-1} = w \}$

the subgroup of S_n that fixes w

So $z_\lambda = |Fix(w)| =$

$= \prod_{i \geq 1} i^{m_i} \, m_i!$



u acts on w by permuting the labels. The only permutations of labels that preserve w are

- cyclic shifts of labels in each cycle of w

corresponds to the factor $\prod_{i \geq 1} i^{m_i} = \lambda_1 \lambda_2 \cdots \lambda_k$

- any permutations of blocks of the same size i

corresp. to the factor $\prod_{i \geq 1} m_i!$

$\square$

Frobenius Formula:

Schur symmetric function

$$\boxed{ch(\chi_\lambda) = s_\lambda}$$

the character of Specht module V_λ

This formula can be reformulated, as follows (the original formulation):

Theorem Let $\lambda = (\lambda_1, \dots, \lambda_k)$, $\lambda \vdash n$ and $\rho = (k-1, k-2, \dots, 0)$. Then

$\chi_\lambda(\nu)$ = the coefficient of the monomial $x^{\lambda+\rho}$ in

$$\left(\prod_{1 \le i < j \le k} (x_i - x_j)\right) p_\nu(x_1, \dots, x_k).$$

Exercise. Show that this Thm $\Leftrightarrow$ $ch(\chi_\lambda) = s_\lambda$.

More general formula for skew shapes.

Theorem. $ch(\chi_{\lambda/\mu}) = s_{\lambda/\mu}$

Usually, people first prove Frobenius formula for χ_λ using representation theory, and then prove the Murnaghan-Nakayama rule using symmetric functions.

But we already proved the M-N rule for $\chi_{\lambda/\mu}$. So in order to prove Frobenius formula it is enough to show that the same M-N rule holds for Schur functions $s_{\lambda/\mu}$.

Murnaghan-Nakayama Rule
(symmetric functions version)

<u>Theorem</u>. Let us expand $s_{\lambda/\mu}$ in the basis of power symmetric functions p_ν:

$$s_{\lambda/\mu} = \sum_{\nu} \frac{\chi_{\lambda/\mu,\nu}}{z_\nu} p_\nu .$$

Then the coefficients $\chi_{\lambda/\mu,\nu}$ are given by

$$\chi_{\lambda/\mu,\nu} = \sum_{\substack{RT \text{ ribbon} \\ \text{tableau of} \\ \text{shape } \lambda/\mu \\ \text{and type } \nu}}' (-1)^{ht(RT)} .$$

First, we prove a related result

<u>Theorem</u>. For any partition μ, and $k \in \mathbb{Z}_{>0}$,

$$\boxed{p_k \cdot s_\mu = \sum_{\substack{\lambda \supseteq \mu \text{ s.t.} \\ \lambda/\mu \text{ is a ribbon} \\ \text{with } k \text{ boxes}}}' (-1)^{ht(\lambda/\mu)} s_\lambda}$$

Corollary (special case for $\mu = \emptyset$)

$$p_k = s_k - s_{(k-1,1)} + s_{(k-2,1^2)} - s_{(k-3,1^3)} + \cdots$$

alternating sum of Schur functions for all hooks with k boxes

$$p_1 = s_{\square}$$

$$p_2 = s_{(2)} - s_{(1,1)}$$

$$p_3 = s_{(3)} - s_{(2,1)} + s_{(1,1,1)}$$

$$p_4 = s_{(4)} - s_{(3,1)} + s_{(2,1,1)} - s_{(1,1,1,1)}$$

etc.

Proof. The ribbons λ/μ that are straight shapes ($\mu = \emptyset$) are exactly hooks $\lambda = (k-r, 1^r)$ $\square$

Let us recall some formulas involving complete homogeneous, elementary, and power symmetric functions (from lecture 2):

$$E(t) := \sum_{k\geq 0} e_k t^k = \prod_{i\geq 1} (1+x_i t)$$

$$H(t) := \sum_{k\geq 0} h_k t^k = \prod_{i\geq 1} \frac{1}{1-x_i t}$$

$$(*) \quad \boxed{H(t)\cdot E(-t) = 1}$$

$$P(t) := \sum_{k\geq 1} p_k t^{k-1}$$

we proved

$$(**) \quad \boxed{\begin{aligned} H'(t) &= P(t)\cdot H(t) \\ E'(t) &= P(-t)\, E(t) \end{aligned}}$$

$(**) \iff$ Newton's formulas for p_k.

Another related formula

$$P(t) \overset{(**)}{=\!=} \frac{H'(t)}{H(t)} \overset{(*)}{=\!=} H'(t) \cdot E(-t)$$

extracting the coefficient of t^{k-1} in both sides we get

Lemma:

$$\boxed{p_k = \sum_{r=1}^{k} r \cdot h_r (-1)^{k-r} e_{k-r}}$$

Examples:

$$p_1 = h_1 = x_1 + x_2 + x_3 + \dots$$

$$p_2 = 2h_2 - h_1 \cdot e_1$$

$$= 2\left((x_1^2 + x_2^2 + \dots) + \sum_{i<j} x_i x_j\right)$$

$$- (x_1 + x_2 + \dots)(x_1 + x_2 + \dots)$$

$$= x_1^2 + x_2^2 + \dots$$

$$p_3 = 3h_3 - 2h_2 \cdot e_1 + h_1 e_2$$

etc.

Let us now prove the formula

$$p_k \cdot s_\mu = \sum_{\substack{\lambda : \\ \lambda/\mu \text{ ribbon} \\ \text{with } k \text{ boxes}}} (-1)^{ht(\lambda/\mu)} s_\lambda$$

combinatorially.

Proof Recall

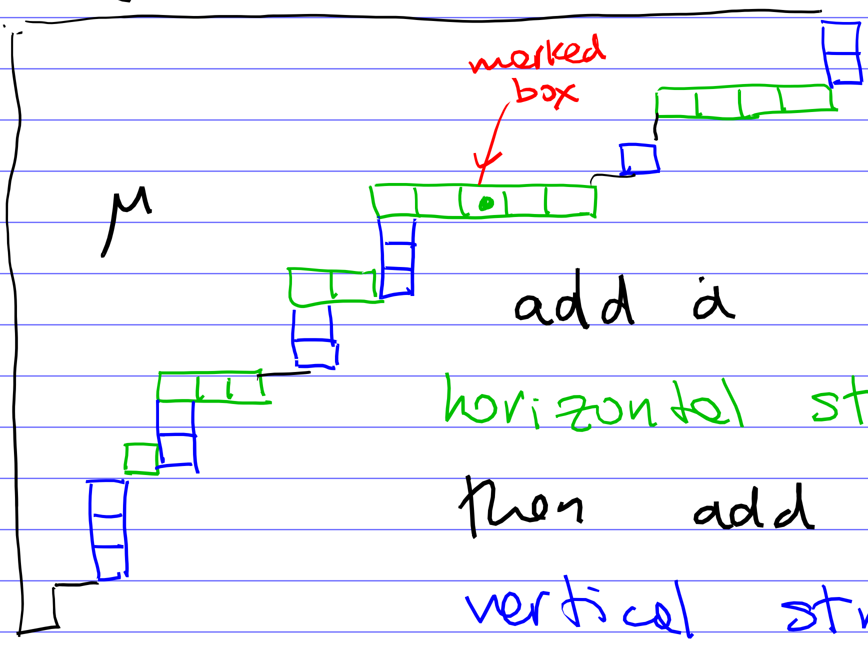
$$h_r \cdot s_\mu = \sum_{\substack{\lambda \supseteq \mu \\ \lambda/\mu \text{ horizontal} \\ r\text{-strip}}} s_\lambda$$

$$e_\ell \cdot s_\mu = \sum_{\substack{\lambda \supseteq \mu \\ \lambda/\mu \text{ vertical} \\ \ell\text{-strip}}} s_\lambda$$

So $p_k \cdot s_\mu = \left(\sum_{\substack{r \geq 1 \\ \ell \geq 0 \\ r+\ell=k}} r (-1)^\ell e_\ell h_r \right) s_\mu$

$$= \sum_{r+\ell=k} \sum_{\substack{\lambda \supseteq \gamma \supseteq \mu \\ \gamma/\mu \text{ horizontal } r\text{-strip} \\ \lambda/\gamma \text{ vertical } \ell\text{-strip}}} r \cdot (-1)^\ell \cdot s_\lambda$$

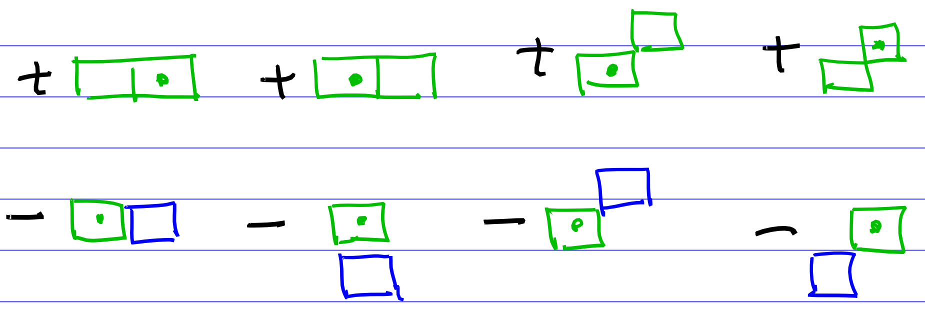
So we first need to add to μ a horizontal r-strip & then add a vertical ℓ-strip. Every such diagram comes with weight $r\cdot(-1)^{\ell}$



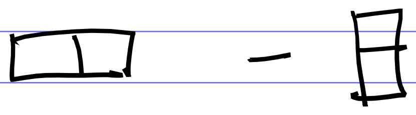
add a horizontal strip then add a vertical strip

- one box in the horizontal strip is marked
  (it account for the factor r)
- Each such diagram comes with the sign $(-1)^{\#\{\text{blue boxes}\}}$

<u>Example</u> All possible types of diagrams for $k=2$



After cancellations, we get only two possible shapes



(ribbons with 2 boxes)

This means that $p_2 S_\mu$ is the sum of S_λ's obtained by adding a horizontal domino to λ minus the sum obtained by adding a vertical domino.

This is exactly what we want to show in general:

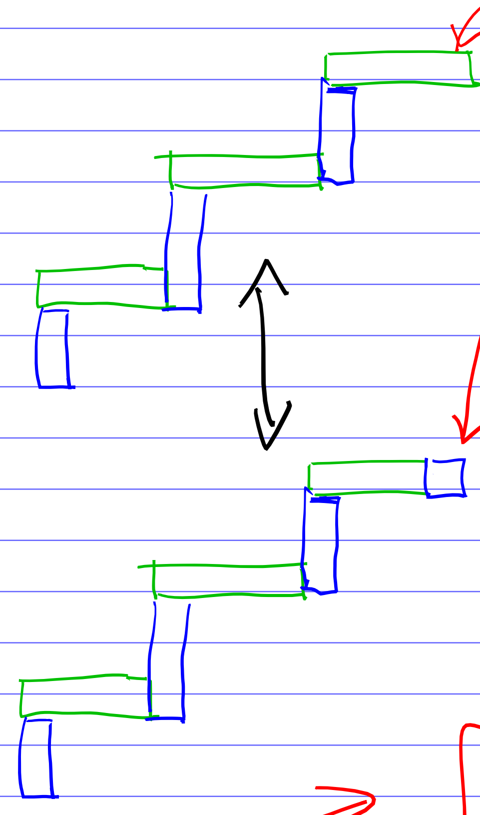
The alternating sum over all diagrams as above = the alternating sum over ribbons.

Let's do this using the "involution principle".

Need: A sign-reversing shape-preserving involution on almost all diagrams as above that cancels all terms, except the terms corresponding to ribbons.

Observation: Any connected component in the diagram has the form:

the rightmost box



We can always switch the color (red $\Leftrightarrow$ blue) of the rightmost box in a connected component (unless it is marked) and get a valid diagram.

we cannot switch the color of the marked box, because it has to be green

Involution on diagrams:

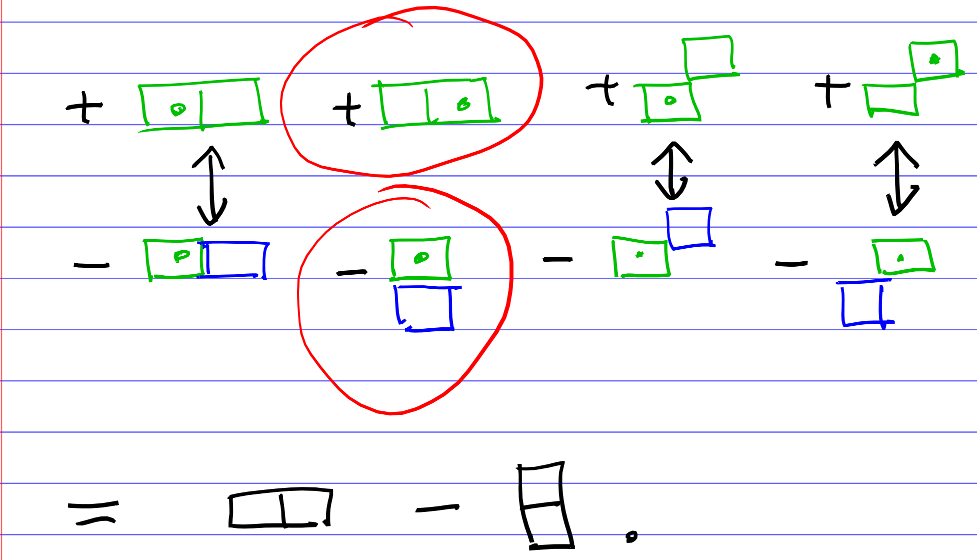
- Find the rightmost connected component s.t. its rightmost box is not marked & reverse the color of this box.

- The involution is not defined only for diagrams s.t.
  - it has exactly 1 connected component, and
  - the rightmost box is marked.

Notice that this transformation

- reverses the sign $= (-1)^{\#\text{blue boxes}}$, i.e. changes the parity to $\#\{\text{blue boxes}\}$
- preserves the shape
- is indeed an involution (if we apply this transformation twice, we get the original diagram)

Example $k=2$



The involution is not defined only for diagrams that consist of a single ribbon (with marked rightmost box).

Notice that $\#\{\text{blue boxes}\}$ is exactly the height of the ribbon. So the sign is

$$(-1)^{\#\{\text{blue boxes}\}} = (-1)^{ht}.$$

So we proved the formula

$$p_k s_\mu = \Big(\sum_{\substack{r+l=k \\ r\geq 1}} r(-1)^l e_l \cdot h_r\Big) s_\mu$$

$$= \sum_{\substack{\lambda:\ \lambda/\mu \\ \text{ribbon with} \\ k \text{ boxes}}}{}' (-1)^{ht(\lambda/\mu)} s_\lambda.$$

$ht(\text{ribbon}) = \#\text{rows} - 1$

1

Applying this formula repeatedly for the product of s_μ with $p_{\nu_1}, p_{\nu_2}, p_{\nu_3}, \ldots$ we get

<u>Theorem</u>

$$p_\nu \cdot s_\mu = \sum_{\substack{\lambda \supseteq \mu \\ |\lambda/\mu| = \nu}}{}' \Big(\sum_{\substack{\text{RT ribbon} \\ \text{tableau of} \\ \text{shape } \lambda/\mu \\ \text{and type } \nu}}{}' (-1)^{ht(RT)} \Big) s_\lambda .$$

this is $\chi_{\lambda/\mu}(\nu)$ by the M-N rule.

But this is <u>not</u> exactly Frobenius formula.

We proved:

(1) $$\boxed{s_\mu \cdot p_\nu = \sum_{\substack{\lambda \supseteq \mu \\ |\lambda/\mu| = |\nu|}} \chi_{\lambda/\mu}(\nu)\, s_\lambda}$$

Need: Frobenius formula.

(2) $$\boxed{s_{\lambda/\mu} = \sum_{\substack{\nu \\ |\nu| = |\lambda/\mu|}} \chi_{\lambda/\mu}(\nu) \frac{p_\nu}{z_\nu}}$$

How to see that (1) $\Leftrightarrow$ (2)?

Orthogonality of Symm. funct.

Hall inner product:

Theorem There exists a unique inner product $\langle \cdot, \cdot \rangle$ on the space Λ of symmetric functions s.t.

- $\boxed{\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}}$

Kronecker delta
$$\delta_{\lambda\mu} = \begin{cases} 1, & \lambda = \mu \\ 0, & \lambda \neq \mu \end{cases}$$

i.e. s_λ's form an orthonormal basis of Λ

- $\boxed{\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}}$

i.e. p_λ form an orthogonal basis of $\Lambda_{\mathbb{R}}$, and

$\{z_\lambda^{-1/2} p_\lambda\}$ forms an orthonormal basis of $\Lambda_{\mathbb{R}}$.

- $\boxed{\langle s_\lambda, f \cdot s_\mu \rangle = \langle s_{\lambda/\mu}, f \rangle}$

for any $f \in \Lambda$.

- $\boxed{\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}}$,

i.e. $\{h_\lambda\}$ $\{m_\lambda\}$ are dual bases of Λ.

Let's show that this thm $\Rightarrow$ $((1) \Leftrightarrow (2))$.

Indeed,

(1) $\Leftrightarrow$ $\chi_{\lambda/\mu}(\nu) = \langle s_\lambda, s_\mu \cdot p_\nu \rangle$.

(2) $\Leftrightarrow$ $\chi_{\lambda/\mu}(\nu) = \langle s_{\lambda/\mu}, p_\nu \rangle$.

How to prove Theorem?

Define the Hall inner product by
$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}.$$

---

this is called the Cauchy product

Lemma. $$\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} =$$

$$= \sum_\lambda s_\lambda(x) s_\lambda(y) \quad \text{(Cauchy's formula RSK)}$$

$$= \sum_\lambda m_\lambda(x) h_\lambda(y)$$ easy to prove just by def. of m_λ & h_λ

$$= \sum_\lambda \frac{1}{z_\lambda} p_\lambda(x) p_\lambda(y)$$ exercise

---

Lemma. Two bases $\{u_\lambda\}$ and $\{v_\lambda\}$ of $\Lambda_{\mathbb{R}}$ are dual w.r.t. $\langle \cdot, \cdot \rangle$ (i.e. $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$)

iff

$$(***) \quad \boxed{\sum_\lambda u_\lambda(x) \cdot v_\lambda(y) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}}$$

Proof. Express
$$u_\lambda = \sum_\gamma a_{\lambda\gamma} h_\gamma$$
$$v_\mu = \sum_\xi b_{\mu\xi} m_\xi$$

$$\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu} \Leftrightarrow \sum_\gamma a_{\lambda\gamma} b_{\mu\gamma} = \delta_{\lambda\mu}$$

$$\Leftrightarrow A B^T = I.$$

$$(***) \Leftrightarrow \sum_\lambda u_\lambda(x) v_\lambda(y) = \sum_\lambda h_\lambda(x) m_\lambda(y)$$
$$\sum_\lambda \Big(\sum_\nu a_{\lambda\nu} h_\nu(x)\Big)\Big(\sum_\gamma b_{\lambda\gamma} m_\gamma(y)\Big)$$

$$\Leftrightarrow \sum_\lambda a_{\lambda\nu} b_{\lambda\gamma} = \delta_{\nu\gamma}$$

$$\Leftrightarrow A^T B = I$$

Now $AB^T = I \Leftrightarrow A^T B = I$ $\square$

It remains to prove

Lemma

$$\langle s_\lambda, s_\mu \cdot f \rangle = \langle s_{\lambda/\mu}, f \rangle \quad \forall f \in \Lambda$$

Example $f = h_1 = x_1 + x_2 + \dots$

LHS:

$$\langle s_\lambda, s_\mu \cdot h_1 \rangle = \langle s_\lambda, \sum_{\substack{\nu \supset \mu \\ \nu/\mu \text{ single box}}} s_\nu \rangle$$

$$= \begin{cases} 1 & \text{if } \lambda/\mu = \square \\ 0 & \text{otherwise} \end{cases}$$

RHS: $\langle s_{\lambda/\mu}, s_\square \rangle =$

$$= \begin{cases} 1 & \text{if } \lambda/\mu = \square \\ 0 & \text{otherwise} \end{cases}$$