18.217 Lecture 21 10/21/2020

<u>last time</u>: V_λ, $\lambda \in \hat{S}_n$, irreducible representations of S_n.

The group algebra $\mathbb{C}[S_n]$, and in particular, the Jucys-Murphy elts.

$$X_i = \sum_{j=1,\dots,i-1} (i,j) \in \mathbb{C}[S_n]$$

$i=1,\dots,n$

also act on V_λ.

<u>Facts:</u>
- Each V_λ has a unique (up to rescaling) basis given by common eigenvectors of the JM-elts. $X_1, \dots, X_n$.
- For each collection of eigenvalues $\alpha = (\alpha_1, \dots, \alpha_n)$, there is at most one basis element v such that $X_i v = \alpha_i v$, for $i=1,\dots,n$.
- These bases are exactly the Gelfand-Tsetlin bases of V_λ's.

$\mathrm{Spec}(n)$:= the set of all vectors $(\alpha_1, \dots, \alpha_n)$ corresponding to basis elements of all V_λ, $\lambda \in \hat{S}_n$

"$\sim$" equiv. relation on $\mathrm{Spec}(n)$

$\alpha \sim \alpha'$ if α and α' correspond to basis elements in the same V_λ.

So basis elements of GT bases can be labelled either by path T in the Bratteli diagram or by vectors $\alpha \in \mathrm{Spec}(n)$:

$$v = v_T = v_\alpha.$$

Theorem. The JM-elts. $X_1, \dots, X_n$ and $s_1, \dots, s_{n-1}$ satisfy the Degenerate Affine Hecke Algebra (DAHA) relations:

- $X_i X_j = X_j X_i \quad \forall\, i,j$
- $s_i X_j = X_j s_i \quad$ if $j \neq i, i+1$
- $s_i X_i = X_{i+1} s_i - 1$
- $s_i X_{i+1} = X_i s_i + 1$

Local Analysis of Spec(n)

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \text{Spec}(n)$ correspond to the basis element $v = v_\alpha = v_T$ in V_λ.

We have

- $\boxed{\alpha_1 = 0}$ (because $X_1 = 0$)
- Suppose $\alpha_i = a$, $\alpha_{i+1} = b$

$$X_i v = a v$$
$$X_{i+1} v = b v$$

Let $v' = s_i(v) \in V_\lambda$

Consider 2 cases:

I. v & v' are linearly dependent

$$s_i^2 = 1 \Rightarrow v' = \pm v$$

DAHA rels. $(s_i X_i = X_{i+1} s_i - 1)$

$$\Rightarrow s_i X_i(v) = X_{i+1} s_i(v) - v$$

$$a \underset{\pm v}{\underset{\|}{v'}} = b \underset{\pm v}{\underset{\|}{v'}} - v$$

$$\Rightarrow \pm a = \pm b - 1 \iff \boxed{b = a \pm 1}$$

II. v and v' are lin. independent

$\Rightarrow$ they span the 2-dim subspace $\langle v, v' \rangle$ in V_λ

The operators X_i, X_{i+1}, s_i act on the subspace $\langle v, v' \rangle$ by matrices:

$$X_i = \begin{pmatrix} a & -1 \\ 0 & b \end{pmatrix}, \quad X_{i+1} = \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix}, \quad s_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X_i v = a v$$

$$X_i v' = X_i s_i v \overset{\text{DAHA rels.}}{=} (s_i X_{i+1} - 1) v$$

$$= b v' - v = -v + b v'$$

So $X_i : \begin{cases} v \mapsto a v + 0 v' \\ v' \mapsto -v + b v' \end{cases}$

Similarly,

$$X_{i+1} : \begin{cases} v \mapsto b v + 0 v' \\ v' \mapsto v + a v' \end{cases}$$

$$s_i : \begin{cases} v \mapsto 0 v + 1 \cdot v' \\ v' \mapsto 1 \cdot v + 0 v' \end{cases}$$

Observation $a \neq b$.

(Otherwise the JM element X_i has a non-trivial Jordan block $\begin{pmatrix} a & -1 \\ 0 & a \end{pmatrix}$. But we know that X_i is diagonalizable $\Rightarrow$ X_i does not have non-trivial Jordan blocks.)

Let's use the basis $\{v, \tilde{v}\}$ in the subspace $\langle v, v' \rangle$, where
$\tilde{v} = v + (b-a) v'$

$X_i : \tilde{v} \mapsto b\tilde{v}$

$X_{i+1} : \tilde{v} \mapsto a\tilde{v}$

$X_j : \tilde{v} \mapsto \alpha_j \tilde{v}$ for $j \neq i, i+1$

the same eigenvalue as vector v

$\Rightarrow$ $\tilde{v}$ is a common eigenvector of $X_1, \dots, X_n$

$\Rightarrow$ $\tilde{v}$ is a (possibly rescaled) element of the GT basis of V_λ

The vector of eigenvalues of $\tilde{v}$ is $\tilde{\alpha} = (\alpha_1, \dots, \underbrace{\alpha_{i+1}, \alpha_i}_{\text{transpose } \alpha_i \,\&\, \alpha_{i+1}}, \dots, \alpha_n)$

In this case $b - a \neq \pm 1$.

(Otherwise, if $\underbrace{b = a \pm 1}$, then contradictio

$\tilde{v} = v \pm s_i(v)$

$\Rightarrow \underbrace{s_i(\tilde{v}) = \pm \tilde{v}}_{\text{So } \tilde{v} \text{ is case I with switched } a \,\&\, b} \overset{(I)}{\Longrightarrow} \overbrace{a = b \pm 1}$.

So we get a contradiction.)

In particular, we obtain

Lemma $\sigma = \sigma_\alpha$ elt. of GT basis

$$s_i(\sigma) = \pm\sigma \iff \alpha_{i+1} = \alpha_i \pm 1$$

Lemma We cannot have

$$(\alpha_1, \dots, \alpha_n) = (\dots\ a, a\pm 1, a, \dots)$$

Proof Suppose

$$\alpha = (\dots\ \overset{i}{a}, \overset{i+1}{a+1}, \overset{i}{a}, \dots)$$

$$\begin{array}{l} s_i\, s_{i+1} \underbrace{s_i\, \sigma}_{} = s_i \underbrace{s_{i+1}\, \sigma}_{} = s_i(-\sigma) = -\sigma \\ \qquad \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad \# \\ s_{i+1}\, s_i \underbrace{s_{i+1}\, \sigma}_{} = s_{i+1}\, s_i(-\sigma) = s_{i+1}(-\sigma) = \sigma \end{array}$$

Contradiction. $\square$

Actually, these conditions uniquely determine the set Spec(n) and "~".

Def. An allowed transposition is $(\alpha_1, \dots, \alpha_n) \leftrightarrow (\alpha_1 \dots \alpha_{i+1} \alpha_i \dots \alpha_n)$ if $\alpha_{i+1} \neq \alpha_i \pm 1$

Theorem For any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathrm{Spec}(n)$ corr. to vector δ in GT-basis, we have

- $\alpha_1 = 0$
- $\alpha_i \neq \alpha_{i+1}$ $\forall$ i
- we cannot have
$$(\alpha_i, \alpha_{i+1}, \alpha_i) = (a, a \pm 1, a)$$
- $\forall$ any allowed transposition $\tilde{\alpha} = (\alpha_1 \dots \alpha_{i+1} \alpha_i \dots)$, we have $\tilde{\alpha} \in \mathrm{Spec}(n)$ and $\tilde{\alpha} \sim \alpha$.

Some corollaries:

Lemma $\forall (\alpha_1, \dots, \alpha_n) \in \mathrm{Spec}(n)$ all $\alpha_i \in \mathbb{Z}$.

Proof. If not, let α_i be the first non-integer eigenvalue. But allowed transposition, we can move α_i to 1st position $\alpha \rightsquigarrow \tilde{\alpha} = (\underset{\alpha_i}{\overset{\tilde{\alpha}_1}{\shortparallel}}, \dots)$

but $\tilde{\alpha}_1 = \alpha_1 = 0$. Contradiction)

Lemma. If $\alpha_i = \alpha_j = a$, $i < j$ then $a-1, a+1 \in \{\alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_{j-1}\}$.

Proof. Suppose not. Find such $\alpha_i = \alpha_j = a$, but $a-1$ or $a \pm 1 \notin \{\alpha_{i+1}, \dots, \alpha_{j-1}\}$, s.t $|j-i|$ is as small as possible & we cannot decrease $|j-i|$ by allowed transpositions.

We have

$$(\alpha_i, \dots, \alpha_j) = (a, \alpha_{i+1}, \dots \alpha_{j-1}, a)$$

$$\alpha_{i+1} = a+1 \text{ or } a-1 \text{ and}$$

$$\alpha_{j-1} = a+1 \text{ or } a-1.$$

If $(\alpha_i \dots \alpha_j) = (a, a+1, \underbrace{\dots,}_{} a+1, a)$

a does not apper here

$\Rightarrow$ we obtain a shorter "bad interval" $(\alpha_{i+1}, \dots, \alpha_{j-1})$. $\square$

Lemma $\forall i > 1$, $\alpha_i - 1$ or $\alpha_i + 1 \in \{\alpha_1, \alpha_2, \dots, \alpha_{i-1}\}$.

Proof. If $\alpha_i = 0$, by previous lemma, both $\alpha_i - 1$ & $\alpha_i + 1$ belong to interval $(\alpha_1 = 0, \alpha_2, \dots, \alpha_i)$

If $\alpha_i \neq 0$ and $\pm 1 \notin \{\alpha_1, \dots, \alpha_i\}$ $\Rightarrow$ by allowed transpositions we can move α_i in 1st position $\alpha \rightsquigarrow \tilde{\alpha} = (\underset{\alpha_i}{\overset{\tilde{\alpha}_1}{\shortparallel}}, \dots)$

But $\tilde{\alpha}_1$ should be 0. $\square$

Define $\text{Cont}(n) \in \mathbb{Z}^n$ as the set of vectors $(\alpha_1, \dots, \alpha_n)$ such that

- $\alpha_1 = 0$
- $\forall\, i > 1$, $\alpha_i - 1$ or $\alpha_i + 1 \in \{\alpha_1, \dots \alpha_{i-1}\}$
- If $\alpha_i = \alpha_j = a$, $i<j$ then $a-1$ and $a+1 \in \{\alpha_{i+1}, \dots \alpha_{j-1}\}$

Let $\approx$ be the equiv. relation on $\text{Cont}(n)$ generated by allowed transpositions.

Previous Theorem says

- $\text{Spec}(n) \subseteq \text{Cont}(n)$.
- For $\alpha, \alpha' \in \text{Spec}(n)$

$$\alpha \approx \alpha' \Rightarrow \alpha \sim \alpha'$$

Let's construct the set $\mathrm{Cont}(n) \subset \mathbb{Z}^n$

$\mathrm{Cont}(1)$: (0)

$\mathrm{Cont}(1)$: $(0, 1)$, $(0, -1)$

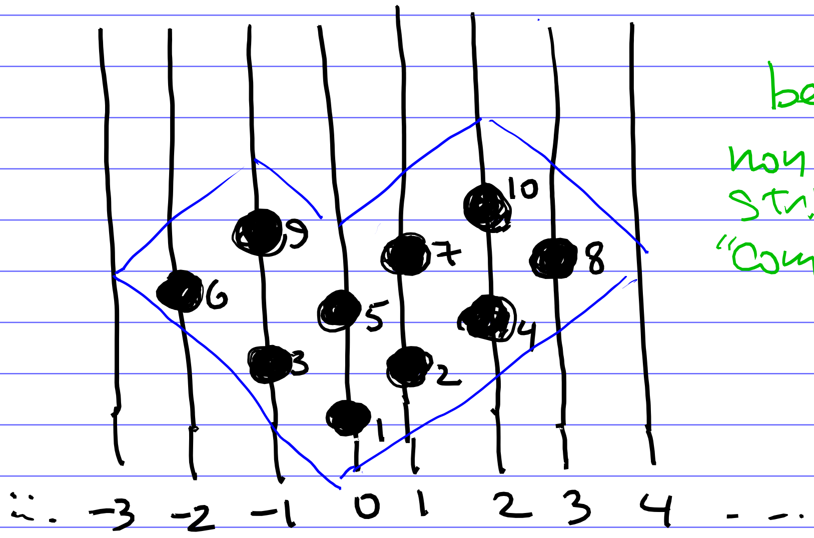
$\mathrm{Cont}(2)$: $(0, 1, 2)$, $(0, 1, -1)$

$$\updownarrow$$

$(0, -1, -2)$, $(0, -1, 1)$

etc.

Let's represent a vector $(\alpha_1, \alpha_2, \dots, \alpha_n)$ by a collection of beads in an <u>abacus</u>



beads on non-adjacent strings can "commute"

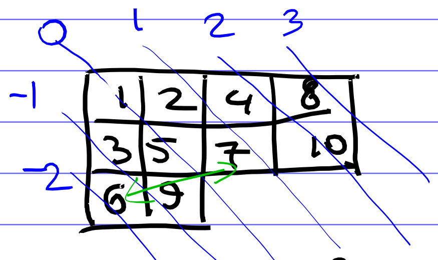
$\rightsquigarrow (\alpha_1, \dots, \alpha_{10}) =$

$= (0, 1, -1, 2, 0, -2, 1, 3, -1, 2)$

Such abaci are just SYT tableaux (in a different orientation)

SYT with n boxes $\rightsquigarrow (\alpha_1, \dots, \alpha_n)$ α_i = the content box filled with i

Previous Example

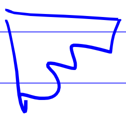


α - content vector of SYT

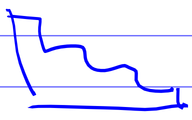
$$\alpha = (\overset{1}{0}, \overset{2}{1}, \overset{3}{-1}, \overset{4}{2}, \overset{5}{0}, \overset{6}{-2}, \overset{7}{1}, \overset{8}{3}, \overset{9}{-1}, \overset{10}{2})$$

an allowed transposition

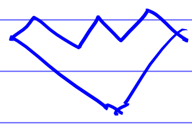
English notation for Young diagrams



French notation



"Russian notation"



Allowed transpositions in a SYT: switches of entries i & $i+1$ if they are located in boxes whose contents differ by ≥ 2.

Lemma Any two SYT's of the same shape can be connected with each other by allowed transpositions.

Theorem $\mathrm{Cont}(n)$ = The set of content vectors of all SYT's with n boxes, $\alpha \approx \tilde{\alpha}$ iff the corresponding Young tableaux have the same shape.

Proof Not hard to prove, for example by induction on n.

Theorem Spec(n) = Cont(n)

"$\sim$" = "$\approx$".

basis vectors $\in$ some irrep

generated by allowed transpositions

Proof We already proved that Spec$(n) \subseteq$ Cont(n)

For $\alpha, \alpha' \in$ Spec(n)

$$\alpha \approx \alpha' \Rightarrow \alpha \sim \alpha'.$$

$\#$ $\sim$-equiv. classes in Spec(n)

$=$ $\#$ irreducible representations of S_n

$=$ $\#$ conjugacy classes in S_n

$= p(n)$ ($\#$ partition of n)

$\#$ $\approx$-equiv. classes in Cont(n)

$=$ $\#$ Young diagrams with n boxes

$= p(n)$

$$\Rightarrow \text{Spec}(n) = \text{Cont}(n)$$

"$\sim$" = "$\approx$"

Corollary Bratteli diagram for $S_0 \subset S_1 \subset S_2 \subset \ldots$ is the Young lattice. $\mathbb{Y}$

Proof We have identified irreps of S_n with $\approx$-equiv. classes of content vectors $\xleftrightarrow{\text{bij}}$ Young diagrams with n boxes.

Branching Rule:

$$\text{Res}^{S_n}_{S_{n-1}} V_\lambda = \bigoplus V_\mu$$

GT-basis for V_λ =

$\bigcup$ GT-bases for V_μ's.

$$\{(\alpha_1, \ldots, \alpha_n)\} \rightsquigarrow \underset{\text{disjoint union}}{\bigcup} \{(\alpha_1, \ldots, \alpha_{n-1})\}$$

vectors of eigenvalues of $X_1, \ldots, X_n$ for GT-basis of V_λ

all vectors of eigenvalues of $X_1, \ldots, X_{n-1}$ for GT-bases of all V_μ's in $\text{Res}^{S_n}_{S_{n-1}} V_\lambda$

Removing α_n from α correspond to removing box filled with n from a SYT. This shows that the branching rule for S_n is given by Young's lattice $\mathbb{Y}$. $\square$

Young's Orthogonal Form

The above construction of irreps. of S_n, can be explicitly presented by matrices.

Now we know $\lambda \in \hat{S_n}$ are identified with Young diagrams.

The GT-basis $\{v_T\}$ of V_λ is labelled by SYT's of shape λ.

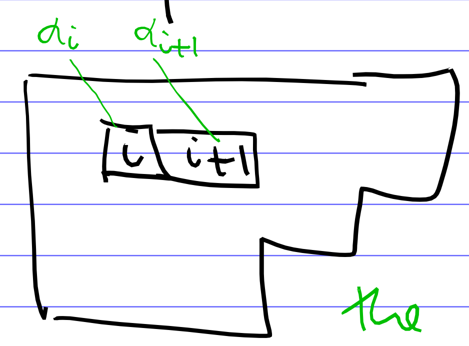
We want to explicitly describe the action of S_n in this basis.

Enough to describe the action of adjacent transpositions $s_1, \dots, s_{n-1}$

Theorem (Young's orthogonal form)

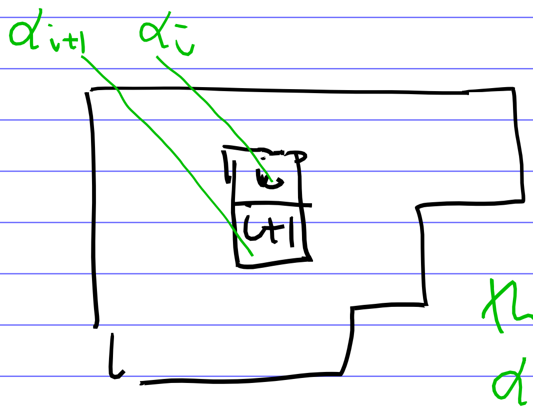
We have

I. $\boxed{s_i : v_T \longmapsto \pm v_T}$ if the entries i & $i+1$ are located in adjacent boxes in the same row / column of T



$s_i : v_T \longmapsto v_T$

the case when $\alpha_{i+1} = \alpha_i + 1$



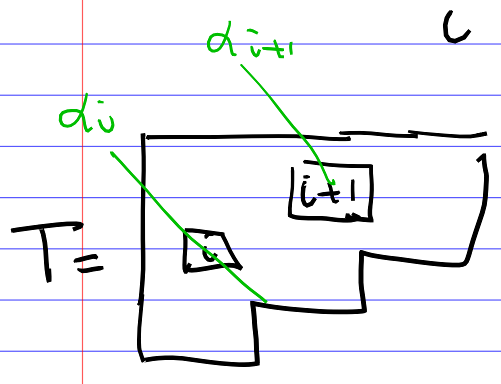
$s_i : v_T \longmapsto -v_T$

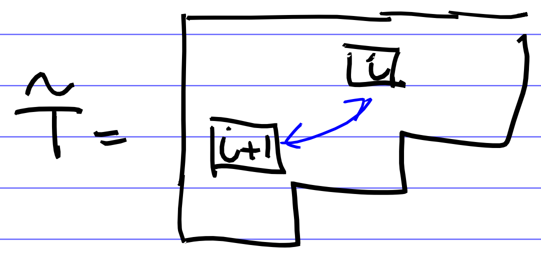
the case when $\alpha_{i+1} = \alpha_i - 1$

II. $\boxed{s_i(v_T) = \frac{1}{d} v_T + \sqrt{1-(\frac{1}{d})^2}\; v_{\tilde{T}}}$

if the entries i & $i+1$ located in non-adjacent boxes of T, d = content of box $(i+1)$ − content of box i.

$\tilde{T}$ = SYT obtained from T by switching i & $i+1$.





The case when $\alpha_{i+1} - \alpha_i \neq \pm 1$

$$s_i : v_T \longmapsto \frac{1}{d} v_T + \sqrt{1-\frac{1}{d^2}}\; v_{\tilde{T}}$$

$d = \alpha_{i+1} - \alpha_i \Leftarrow$ the difference of contents of $i+1$ & i.

$T \rightsquigarrow \tilde{T}$ is an allowed transposition

Proof We already considered these two cases in our local analysis of $(\alpha_1, \dots, \alpha_n)$

I. $s_i v_T = \pm v_T$

$$\Updownarrow$$

$$\alpha_{i+1} = \alpha_i \pm 1$$

II. v_T & $s_i(v_T)$ are lin. indep. ($\iff \alpha_{i+1} - \alpha_i \neq \pm 1$)

Then $\tilde{T}$ corresponds to allowed transposition

$$\alpha = (\alpha_1 \dots \alpha_n)$$
$$\tilde{\alpha} = (\alpha_1 \dots \alpha_{i+1} \alpha_i \dots \alpha_n)$$

Then vectors v_T, $v_{\tilde{T}}$ from the GT-basis are exactly the vector v and the rescaling of vector $\tilde{v}$ (from local analysis) s.t $|v| = |\tilde{v}| = 1$.

If we express $s_i(v)$ in v_T & $v_{\tilde{T}}$ we obtain needed formules.

Example $n=3$ $\lambda = \ydiagram{2,1}$

2 SYT's $\ytableaushort{12,3}$ $\ytableaushort{13,2}$

$$S_1 : v_{\ytableaushort{12,3}} \longmapsto v_{\ytableaushort{12,3}}$$

$$S_1 : v_{\ytableaushort{13,2}} \longmapsto -v_{\ytableaushort{13,2}}$$

$$S_2 : v_{\ytableaushort{12,3}} \longmapsto \frac{1}{-2} v_{\ytableaushort{12,3}} + \sqrt{1-\frac{1}{4}}\, v_{\ytableaushort{13,2}}$$

$$S_2 : v_{\ytableaushort{13,2}} \longmapsto \frac{1}{2} v_{\ytableaushort{13,2}} + \sqrt{1-\frac{1}{4}}\, v_{\ytableaushort{12,3}}$$

We can use Young's orthogonal form to explicitly calculate the <u>characters</u> of irreducible representations

---

<u>Characters of irreps of S_n</u>

$$R : G \mapsto GL(V)$$

Its characters is

$$\chi_R : g \in G \mapsto tr(R_g)$$

$\chi_R : G \to \mathbb{C}$ is a class function on S_n

<u>Example</u> The character table of S_3

	conj class / irrep	id	$(1,2)$	$(1,2,3)$
trivial rep	$V_{\begin{array}{\|c\|c\|c\|}\hline \, & \, & \, \\\hline\end{array}}$	1	1	1
sign rep	$V_{\begin{array}{\|c\|}\hline \, \\\hline \, \\\hline \, \\\hline\end{array}}$	1	-1	1
2-dim rep	$V_{\begin{array}{\|c\|c\|}\hline \, & \, \\\hline \, \\\cline{1-1}\end{array}}$	2	0	-1

$$s_1 : \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \longmapsto \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \longmapsto -\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$$

$$\chi_{\begin{array}{|c|c|}\hline \, & \, \\\hline \, \\\cline{1-1}\end{array}}(s_1) = \operatorname{trace}(\uparrow) = 1-1 = 0$$

$$s_2 : \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \mapsto -\frac{1}{2}\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} + \dots$$

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \longmapsto \frac{1}{2}\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} + \dots$$

$$\chi_{\begin{array}{|c|c|}\hline \, & \, \\\hline \, \\\cline{1-1}\end{array}}(s_2 s_1) = \operatorname{trace}(s_2 s_1)$$

$$= 1 \cdot \left(-\frac{1}{2}\right) + (-1)\left(\frac{1}{2}\right)$$

$$= -1$$