18.217 Lecture 20 10/19/2020

We'll continue with Vershik-Okounkov construction.

<u>Last time</u>: G any finite group

- G has finitely many (up to isomorphism) irreducible representations V_λ, $\lambda \in G^\wedge$.
  $|G^\wedge| = \#$ conjugacy classes in G
- <u>Group algebra</u> of G
  $$\mathbb{C}[G] = \{\sum_{g\in G} f_g\, g\} \qquad f_g \in \mathbb{C}$$
  $$\simeq \left\{ \begin{bmatrix} \overbrace{\boxed{}}^{d_1} & & & 0 \\ & \overbrace{\boxed{}}^{d_2} & & \\ & & \ddots & \\ 0 & & & \overbrace{\boxed{}}^{d_N} \end{bmatrix} \right\}$$
  algebra of block diagonal matrices / $\mathbb{C}$ (with square blocks)

block sizes $d_1, \dots, d_N$ are $\dim(V_\lambda)$
(Here we assume $G^\wedge = \{1, \dots, N\}$
$N = |G^\wedge|$.)

Explicitly: Pick linear bases in all V_λ's. Then the represent. V_λ is given by homomorphism:
$$R_\lambda : G \to GL_{d_\lambda}$$
$d_\lambda \times d_\lambda$ matrices w/ $\det \neq 0$.

$$\begin{array}{l} g \mapsto R(g) = \begin{bmatrix} \boxed{R_1(g)} & & & 0 \\ & \boxed{R_2(g)} & & \\ & & \ddots & \\ 0 & & & \boxed{R_N(g)} \end{bmatrix} \\ \cap \\ G \\ \cap \\ \mathbb{C}[G] \end{array}$$

This linearly extends to the map $\mathbb{C}[G] \to \left\{ \begin{array}{c} \text{block-diagonal} \\ \text{matrices} \end{array} \right\}$

$$\sum_{g\in G} f_g\, g \longmapsto \sum_{g\in G} f_g\, R(g)$$

• The <u>center</u> of the group algebra

$$Z_{\mathbb{C}[G]} := \{ f \in \mathbb{C}[G] \mid fg = gf \ \ \forall g \in G \}$$

$$= \mathbb{C}_{\text{class}}(G) := \left\{ \sum_{g\in G} f_g \cdot g \;\middle|\; g \mapsto f_g \text{ is a } \underline{\text{class}} \ \underline{\text{function on } G} \right\}$$

algebra of class functions.

const. on conj. classes

$$\simeq \left\{ \begin{bmatrix} \begin{matrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_1 \end{matrix} & & & 0 \\ & \begin{matrix} \alpha_2 & & 0 \\ & \ddots & \\ 0 & & \alpha_2 \end{matrix} & & \\ & & \ddots & \\ 0 & & & \begin{matrix} \alpha_N & & \\ & \ddots & \\ & & \alpha_N \end{matrix} \end{bmatrix} \right\}$$

$$\alpha_1, \ldots, \alpha_N \in \mathbb{C}$$

• $G_0 = \{1\} \subset G_1 \subset G_2 \subset \ldots$

any sequence of included groups.

– Bratteli diagram: directed graph on vertex set $\bigcup_{i\geq 0} \hat{G_i}$

edges correspond to "branching rule" $\operatorname{Res}^{G_n}_{G_{n-1}} V_\lambda = \bigoplus V_\mu$

$$\underset{\hat{G}_{n-1}}{\mu} \xrightarrow{m} \underset{\hat{G}_n}{\lambda} \quad \text{if } V_\mu \text{ has appears in } \operatorname{Res}^{G_n}_{G_{n-1}} V_\lambda$$

with multiplicity m.

- $\dim V_\lambda$ = # directed paths $T = (\emptyset \to \dots \to \lambda)$ in the Bratteli diagram

(Here $\hat{G_0} = \{\emptyset\}$.)

- Gelfand-Tsetlin basis of V_λ

$$\{ v_T \mid T = (\emptyset \to \dots \to \lambda) \}$$

$$V_\lambda \overset{\mathrm{Res}^{G_n}_{G_{n-1}}}{\rightsquigarrow} \bigoplus_{\mu \in \hat{G}_{n-1}} V_\mu \overset{\mathrm{Res}^{G_{n-1}}_{G_{n-2}}}{\rightsquigarrow} \bigoplus \left(\bigoplus_{\nu \in \hat{G}_{n-2}} V_\nu \right)$$

$\lambda \in \hat{G_n}$

$$\rightsquigarrow \dots \rightsquigarrow V_\lambda = \bigoplus_T \{\text{one-dimensional spaces}\}$$

Then pick a generator v_T in each 1-dim space.

- If Bratteli diagram does not have multiple edges then a Gelfand-Tsetlin basis is unique up to rescaling the vectors v_T.

$$G_0 \subset G_1 \subset G_2 \subset \cdots$$

$$\begin{array}{ccccccc} \mathbb{C}[G_0] & \subset & \mathbb{C}[G_1] & \subset & \mathbb{C}[G_2] & \subset & \cdots \\ \cup & & \cup & & \cup & & \\ Z_0 & & Z_1 & & Z_2 & & \end{array}$$

$Z_n := Z_{\mathbb{C}[G_n]}$ the center of group alg.

- <u>Gelfand-Tsetlin</u> <u>subalgebra</u>

$GT_n \subset \mathbb{C}[G_n]$

GT_n = algebra generated by $Z_0, Z_1, Z_2, \ldots$

<u>Proposition</u> If we pick a GT-basis in each V_λ. Then

$$GT_n \simeq \left\{ \begin{matrix} \text{all diagonal} \\ \text{matrices} \end{matrix} \right\} \subset \left\{ \begin{bmatrix} \boxed{\diagdown\diagdown} & & 0 \\ & \boxed{\diagdown\diagdown} & \\ 0 & & \ddots \ \boxed{\diagdown\diagdown} \end{bmatrix} \right\}$$

$$\text{IS} \quad Z_n$$

In particular, GT_n is a maximal commutative subalgebra of $\mathbb{C}[G_n]$.

- The <u>centralizer</u> of $\mathbb{C}[G_{n-1}]$ in $\mathbb{C}[G_n]$

$$Z_{n-1,1} := \{ f \in \mathbb{C}[G_n] \mid f g = g \cdot f \ \forall g \in G_{n-1} \}$$

all elements of $\mathbb{C}[G_n]$ that commute with all elements of $\mathbb{C}[G_{n-1}]$

<u>Proposition</u> The Brotteli diagram does not have multiple edges iff all centorizers $Z_{n-1,1}$ are commutative.

In this case,

- GT-basis $\{v_T\}$ is unique (up to rescaling of v_T's)
- a vector $v \in V_\lambda$ belongs to GT-basis (up to resc.) iff v is a common eigenvector of all element of GT_n
- Basis elements in GT-basis are uniquely determined by <u>eigenvalues</u> of elements of GT_n

We will not prove above Proposition. Proofs can be found in the paper by Vershik-Okounkov.
They are not very hard.
This is basically linear algebra.

Let's now specialize the above general construction to the case of symmetric groups S_n included into each other is the standard way:

$$S_0 \subset S_1 \subset S_2 \subset S_3 \subset \dots$$

(Young) - Jucys - Murphy elements: For $i = 1, 2, \dots$

$$X_i := (1, i) + (2, i) + \dots + (i-1, i).$$

$X_1 = 0$

$X_2 = (1, 2)$

$X_3 = (1, 3) + (2, 3)$

$X_4 = (1, 4) + (2, 4) + (3, 4)$

etc.

Recall, $Z_n := Z_{\mathbb{C}[S_n]}$

$Z_{n-1,1}$ = centralizer of $Z_{\mathbb{C}[S_{n-1}]}$ in $Z_{\mathbb{C}[S_n]}$

GT_n = subalg. of $\mathbb{C}[S_n]$ generated by $Z_1, Z_2, \dots, Z_n$

$$Z_n \subset Z_{n-1,1} \subset GT_n \subset \mathbb{C}[S_n]$$

Clearly, $X_n = \sum$ all transp in S_n $- \sum$ all transp. in S_{n-1} $\in \langle Z_n, Z_{n-1} \rangle \subset GT_n$.

<u>Theorem</u>. GT_n is the algebra generated by $X_1, X_2, \dots, X_n$:

$$GT_n = \langle X_1, X_2, \dots, X_n \rangle.$$

This follows from.

<u>Theorem</u>. $Z_{n-1,1}$ is generated by Z_{n-1} and X_n:

$$Z_{n-1,1} = \langle Z_{n-1}, X_n \rangle.$$

In particular, $Z_{n-1,1}$ is commutative. Thus the Bratteli diagram does not have multiple edges.

(It is easy to see that X_n commutes with Z_{n-1}.)

This claim can by proved by induction.

For a partition $(c_1, \dots, c_\ell)$

$n = c_1 + \dots + c_\ell$

Let $[c_1, \dots, c_\ell] := \sum\limits_{\substack{w \in S_n \\ \text{with cyclic type} \\ c_1, \dots, c_\ell}} w \in \mathbb{C}[S_n]$

$w = \underbrace{(\dots)}_{c_1} \underbrace{(\dots)}_{c_2} \dots \underbrace{(\dots)}_{c_\ell}$

A <u>marked partition</u> is a partition with one marked part $n = \bar{c}_1 + \dots + c_\ell$

(c_1 non necessarily the largest part)

$[\bar{c}_1, \dots, c_\ell] := \sum\limits_{\substack{w \in S_n \\ \text{with cyclic} \\ \text{type } c_1, \dots, c_\ell \\ \text{s.t. cycle of size } c_1 \\ \text{contains 'n'}}} w \in \mathbb{C}[S_n]$

<u>Example</u> $[2, \underbrace{1, \dots, 1}_{n-2}] =$

= the sum of all transpositions in S_n.

$[\bar{2}, \underbrace{1, \dots, 1}_{n-2}] =$ the sum of all transp. (i, n) in S_n

$=: X_n.$

Easy lemma:

- Z_n has linear basis $[c_1, \dots, c_\ell]$ (given by all partitions of n)
- $Z_{n-1,1}$ has linear basis $[\bar{c}_1, \dots, c_\ell]$ (given by all marked partitions of n)

(This directly follows from definitions of Z_n, $Z_{n-1,1}$.)

All above claims about JM-elts follow from

Lemma. All $[\bar{c}_1, \dots, c_\ell]$ ($c_1 + \dots + c_\ell = n$), can be expressed in terms of

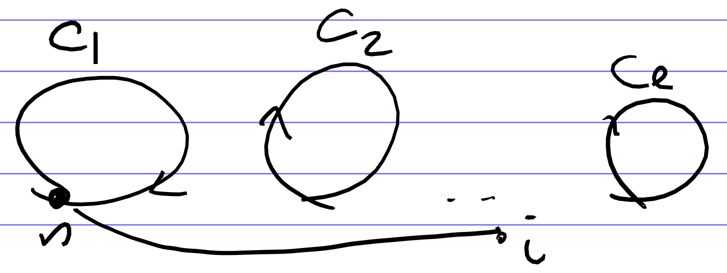
marked partitions of n

$$[\tilde{c}_1, \dots, \tilde{c}_\ell] \quad (\tilde{c}_1 + \dots + \tilde{c}_\ell = n-1)$$

unmarked partitions of $n-1$

and X_n.

Proof. $X_n \cdot [\bar{c}_1, c_2, \dots, c_\ell] = ?$



2 cases → $i \in$ marked cycles c_1
→ $i \in$ some other cycles c_j

$X_n [\bar{c}_1, \dots, c_\ell] =$

marked cycle splits into 2 cycles

$$= \sum_{c_1' + c_1'' = c_1} \begin{pmatrix}\text{some}\\ \text{non-zero}\\ \text{coeff.}\end{pmatrix} [\bar{c}_1', c_1'', c_2 \dots, c_\ell]$$

$$= \sum_{j \in \{2, \dots, \ell\}} \begin{pmatrix}\text{some}\\ \text{non-zero}\\ \text{coeff}\end{pmatrix} [\overline{c_1 + c_j}, c_2, \dots \hat{c}_j \dots c_\ell]$$

the marked cycle merges with some other cycle

One can use this identity to prove the lemma by induction. □

Exercise. Do this.

Lemma $\Rightarrow$ Each $[c_1, \dots, c_\ell]$ $(c_1 + \dots + c_\ell = n)$ can be expressed in terms of $X_1, X_2, \dots, X_n$

$\Leftrightarrow$ each element in Z_n can be expressed in $X_1, \dots, X_n$.

$\Leftrightarrow$ $GT_n = \langle X_1, \dots, X_n \rangle$.

Examples

$$[2, \underbrace{1 \dots 1}_{n-2}] = X_1 + \dots + X_n$$

$$[3, \underbrace{1, \dots, 1}_{n-3}] = X_1^2 + \dots + X_n^2 - \binom{n}{2}.$$

etc.

Now we have:

Corollary Each irreducible representation V_λ of S_n has a unique (up to rescaling) basis $\{v_T\}$ (GT-basis) such that the vectors v_T are the common eigenvectors of JM-elements $X_1, \dots, X_n$

Each basis vector v_T is uniquely determined by the collection $(\alpha_1, \dots, \alpha_n)$ of eigenvalues

$$X_i v_T = \alpha_i v_T.$$

Elements v_T of GT-basis can be labelled by

- paths T in the Bratteli diagr.
- vectors $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ of eigenvalues of $X_1, \dots, X_n$.

$$T \Longleftrightarrow (\alpha_1, \dots, \alpha_n)$$

Let $\mathrm{Spec}(n) :=$ the set of all possible vectors $(\alpha_1, \dots, \alpha_n)$ for all irreps of S_n

Equiv. relation

$(\alpha_1, \dots, \alpha_n) \sim (\alpha_1', \dots, \alpha_n')$ if

$(\alpha_1, \dots, \alpha_n)$ and $(\alpha_1', \dots, \alpha_n')$ corresp. to vectors v_T, $v_{T'}$ in the GT-basis of the same irreducible representation V_λ.

<u>Our goal</u> is to describe $\mathrm{Spec}(n)/\sim$ combinatorially

Our main tool is

Theorem The elements $s_1, \dots, s_{n-1}$ and $X_1, \dots, X_n$ in $\mathbb{C}[S_n]$ satisfy the relations:

- Coxeter rels. for $s_1, \dots, s_{n-1}$
- $X_i X_j = X_j X_i, \ \forall i, j$
- $s_i X_j = X_j s_i$, if $j \neq i, i+1$
- $s_i X_i = X_{i+1} s_i - 1$
- $s_i X_{i+1} = X_i s_i + 1$.

Proof Easy verification.

Def. The algebra with above relations is called

Degenerate Affine Hecke Algebra (DAHA).

Local Analysis of Spec(u).

$(\alpha_1, \ldots, \alpha_n) \in \mathrm{Spec}(u)$

corresp. to basis vector $v = v_T$ in V_λ

- $\alpha_1 = 0$ (because $X_1 = 0$)

Suppose $\alpha_i = a$, $\alpha_{i+1} = b$

$X_i v = a v$,

$X_{i+1} v = b v$

Let $v' = s_i(v) \in V_\lambda$

Consider 2 cases:

I. v & v' are lin. dependent

$s_i^2 = 1 \Rightarrow v' = \pm v.$

DAHA rels. $\Rightarrow s_i X_i + 1 = s_i X_{i+1}$

apply this to eigenvector v.

$\pm a v + v = \pm b v$

$$\boxed{b = a \mp 1}$$

II v, v' are linearly independent $\Rightarrow$ they span 2-dim subspace in V_λ.

The operators X_i, X_{i+1}, s_i act on the subspace $\langle v, v' \rangle$ by matrices:

$$X_i = \begin{pmatrix} a & -1 \\ 0 & b \end{pmatrix}, \quad X_{i+1} = \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix}, \quad s_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$X_i v = a_i v$$

DAHA relation

$$X_i v' = X_i s_i(v) = (s_i X_{i+1} - 1) v$$

$$= -v + b v'.$$

etc.

<u>Observation</u> $a \neq b$

(Otherwise X_i has a non-trivial Jordan block $\begin{bmatrix} a & -1 \\ 0 & a \end{bmatrix}$ but we know that X_i is diagonalizable.

Let's use the basis
$\{v, \tilde{v}\}$ insead of $\{v, v'\}$
where $\tilde{v} = v + (b-a)v'$.

$X_i : \tilde{v} \mapsto b\tilde{v}$

$X_{i+1} : \tilde{v} \mapsto a\tilde{v}$ the same eigenvalue as vector v

and $X_j \tilde{v} = \alpha_j \tilde{v}$
for any $j \neq i, i+1$

$\Rightarrow \tilde{v}$ is a common eigenvector of $X_1, \dots X_n$

$\Rightarrow \tilde{v} \in$ GT basis
(of the same irrep V_λ)

The vector of eigenvalues of $\tilde{v}$
is $\tilde{\alpha} = (\alpha_1, \dots, \underbrace{\alpha_{i+1}, \alpha_i}, \dots \alpha_n)$

transpose α_i and α_{i+1} in vector α.

If $b = a \pm 1$ then $s_i \tilde{v} = \pm \tilde{v}$
$\Rightarrow \tilde{v}$ & $s_i(\tilde{v})$ lin. dep.
$\Rightarrow v$ & $s_i(v)$ lin. dep.
But we assume that v & $s_i(v)$
are indep
So we cannot have $b = a \pm 1$
in this case.

We obtain

<u>Theorem</u>. Let $\alpha = (\alpha_1, .., \alpha_n) \in Spec(n)$ corresp. to vector v_T in GT-basis

- $\alpha_1 = 0$
- $\alpha_i \neq \alpha_{i+1} \quad \forall i$
- if $\alpha_{i+1} = \alpha_i \pm 1$

then $s_i(v_T) = \mp v_T$

- if $\alpha_{i+1} \neq \alpha_i \pm 1$

then $\tilde{\alpha} = (\alpha_1, \dots, \alpha_{i+1}, \alpha_i, \dots \alpha_n) \in Spec(n)$

$\tilde{\alpha} \sim \alpha$

$\tilde{\alpha}$ corresp. to basis vector $v_{\tilde{T}}$

s_i preserves the 2-dim subspace $\langle v_T, v_{\tilde{T}} \rangle$

- We cannot have

$\alpha = (\alpha_1, \dots \alpha_n) = (\dots a, a \pm 1, a, \dots)$

<u>Proof</u> We already proved everything exept the last claim

Last claim: Suppose

$$\alpha = (\dots \overset{i}{a}, \overset{i+1}{a+1}, \overset{i}{a} \dots)$$

$$s_i s_{i+1} s_i (v_T) = - v_T$$

$$\parallel$$

$$s_{i+1} s_i s_{i+1} (v_T) = v_T$$

Contradiction. $\square$

Claim The above conditions uniquely describe the set Spec(n) & equiv. rel. $\sim$.

Let's be more specific.

Define An allowed transposition as $(\alpha_1 \dots \alpha_n) \longleftrightarrow (\alpha_1 \dots \alpha_{i+1} \alpha_i \dots \alpha_n)$ if $\alpha_{i+1} \neq \alpha_i \pm 1$.

Define $\mathrm{Cont}(n) \in \mathbb{Z}^n$ the set of vectors $(\alpha_1, \dots, \alpha_n)$ such that

- $\alpha_1 = 0$
- If $\alpha \leftrightarrow \tilde{\alpha}$ is an allowed transposition then $\tilde{\alpha} \in \mathrm{Cont}(n)$
- If $\alpha_i = \alpha_j = a$, $i<j$, then $a-1, a+1 \in \{\alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_{j-1}\}$

Let $\approx$ be an equiv. rel. on $\mathrm{Cont}(n)$ generated by allowed transpositions.

We proved that

$$\mathrm{Spec}(n) \subseteq \mathrm{Cont}(n)$$

$$\alpha \sim \alpha' \implies \alpha \approx \alpha'$$

<u>Theorem</u> The set $\mathrm{Cont}(n)$ is in bijection with stand. Young tableaux of some shape $\lambda \vdash n$

$\alpha \approx \alpha'$ if corresp. SYT's have the same shape

So # $\approx$-equiv. classes $= p(n)$ (# Young tableaux $\lambda \vdash n$)

On the other hand.

$\#\{\sim$-equiv. classes in $\mathrm{Spec}(n)\}$

$=$ # irreps of S_n $= p(n)$

So we get

<u>Theorem</u> $\mathrm{Spec}(n)/\sim$

$= \mathrm{Cont}(n)/\approx$.