18.217 Combinatorial Theory

Symmetric Group & Symmetric Funct.

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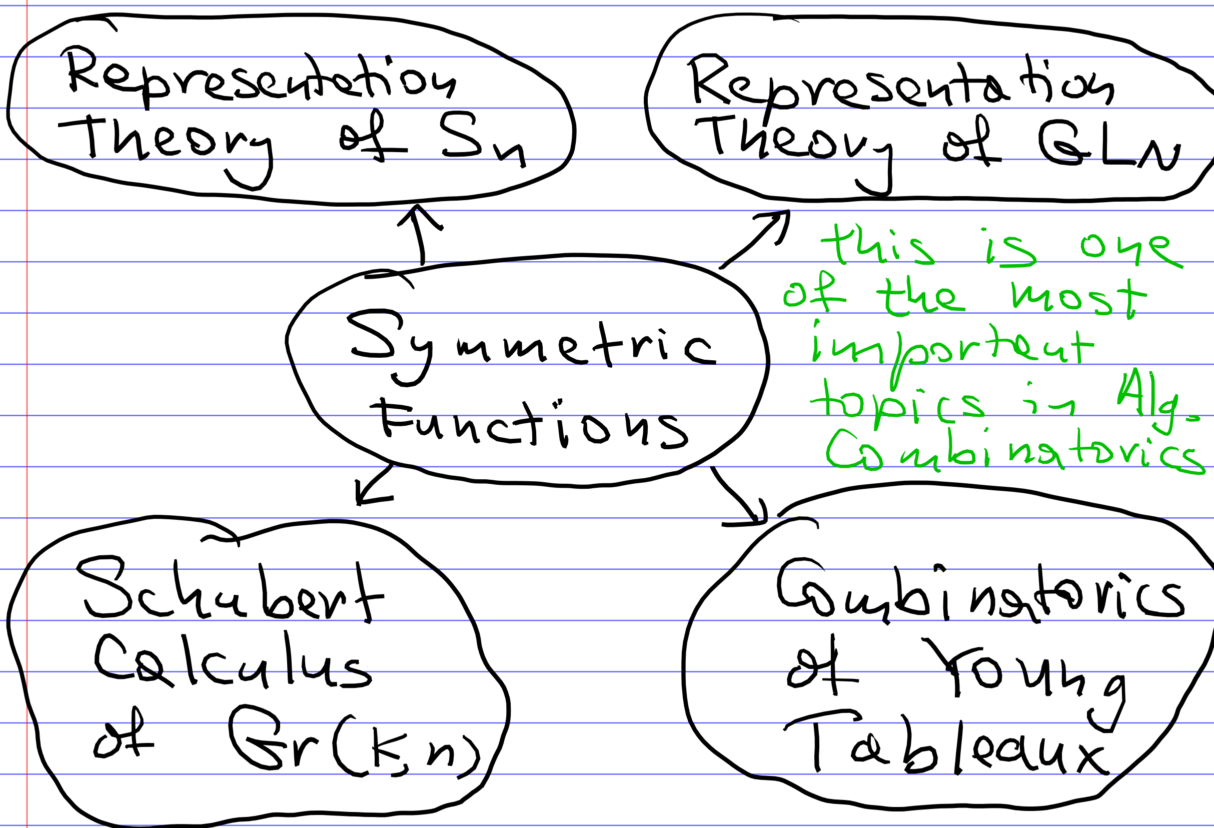


We'll follow terminology and notations from [Macdonald] "Symm. Funct. and Hall Polyn."

S_n : symmetric group of permutations of $1, 2, \ldots, n$.

Its elements are permutations $w : [n] \xrightarrow{\text{bijective maps}} [n]$

where $[n] := \{1, 2, \ldots, n\}$.

1-line notation for permutations: $w = w_1, \ldots, w_n$

2-line notation: $w = \begin{pmatrix} 1 & 2 & \ldots & n \\ w_1 & w_2 & \ldots & w_n \end{pmatrix}$

$w_i = w(i)$, $i \in [n]$.

The ring of symmetric polynomials

$$\Lambda_n := \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$$

Its elements are polynomials $f(x_1, \ldots, x_n)$ s.t.

$f(x_1, \ldots, x_n) = f(x_{w_1}, \ldots, x_{w_n})$, for any permutation $w \in S_n$

i.e. they are S_n-inveriant polyn.

Examples: $x_1 + \dots + x_n$

$x_1^2 + \dots + x_n^2$, $x_1x_2 + x_1x_3 + \dots + x_{n-1}x_n$,

$(x_1 + \dots + x_n)^2$, etc.

Λ_n is a graded ring:

$$\Lambda_n = \Lambda_n^0 \oplus \Lambda_n^1 \oplus \Lambda_n^2 \oplus \dots$$

Λ_n^k consists of homogeneous symm. polynomials of degree k (together with 0).

For example: $x_1 + \dots + x_n \in \Lambda_n^1$

$x_1^2 + \dots + x_n^2 \in \Lambda_n^2$, etc.

Symmetric Functions

as opposed to polynomials

$$\Lambda^k := \varprojlim \Lambda_n^k$$

inverse limit under the projections $\Lambda_{n+1}^k \to \Lambda_n^k$

$$f(x_1, \dots, x_{n+1}) \to f(x_1, \dots, x_n, 0)$$

In other words,

Λ^k consists of all homogeneous degree k power series in infinitely many variables $x_1, x_2, x_3, \ldots$ which are invariant under all permutations of x_i's (together with 0)

<u>Example</u>: $e_1 = x_1 + x_2 + \ldots \in \Lambda^1$

$p_2 = x_1^2 + x_2^2 + \ldots \in \Lambda^2$

these are <u>not</u> polynomials because they are given by <u>infinite</u> sums.

Finally, the

<u>ring of symmetric functions</u>

is $\Lambda := \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \ldots$

Equivalently, Λ consists of all formal power series f in $x_1, x_2, x_3, \ldots$ (inf. many x_i's) with integer coefficients s.t.

(1) $f(x_1, x_2, x_3, \ldots)$ is invariant under all permutations of x_i's

(2) f has bounded degree.

Ex. $1 + x_1 + x_2 + \ldots \in \Lambda$

$$\sum_{\substack{i \geq 1 \\ k \geq 1}} x_i^k \notin \Lambda$$

degrees are not bounded

Some elements of Λ

$x_1 + x_2 + \ldots$, $\quad x_1^2 + x_2^2 + \ldots$

$x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + x_3^2 x_1 + \ldots$

all monomials obtained from $x_1^2 x_2$ by permutations of the variables.

More generally,

Take a <u>partition</u> λ of k

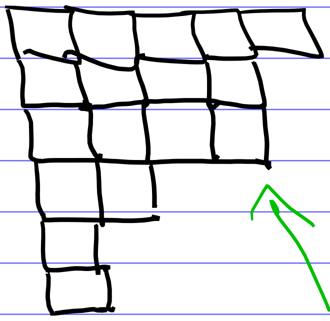
$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_e)$

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_e > 0$ integers

$\lambda_1 + \dots + \lambda_e = k$

Notation $\lambda \vdash k$.

We'll identify a partition λ with its <u>Young diagram</u>

$\lambda =$ 

λ_1 boxes
λ_2 boxes
λ_3 boxes
$\dots$

this particular Young diagr. corresponds to $\lambda = (5, 4, 4, 2, 1, 1)$

<u>Monomial symmetric functions:</u>

$$m_\lambda := x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_e^{\lambda_e} + \left(\text{all other monomials obtained by perms of the variab}\right)$$

for a partition λ

Examples

$$m_{(1)} = m_{\square} = x_1 + x_2 + x_3 + \dots$$

$$m_{(2)} = m_{\square\square} = x_1^2 + x_2^2 + x_3^2 + \dots$$

$$m_{(2,1)} = m_{\substack{\square\square \\ \square}} = x_1^2 x_2 + x_2^1 x_1 + \dots$$

etc.

Clearly, $\{m_\lambda \mid \lambda \vdash k\}$ forms a linear basis of Λ^k. So we get.

Fact: $\dim \Lambda^k = p(k)$ the number of partitions of k

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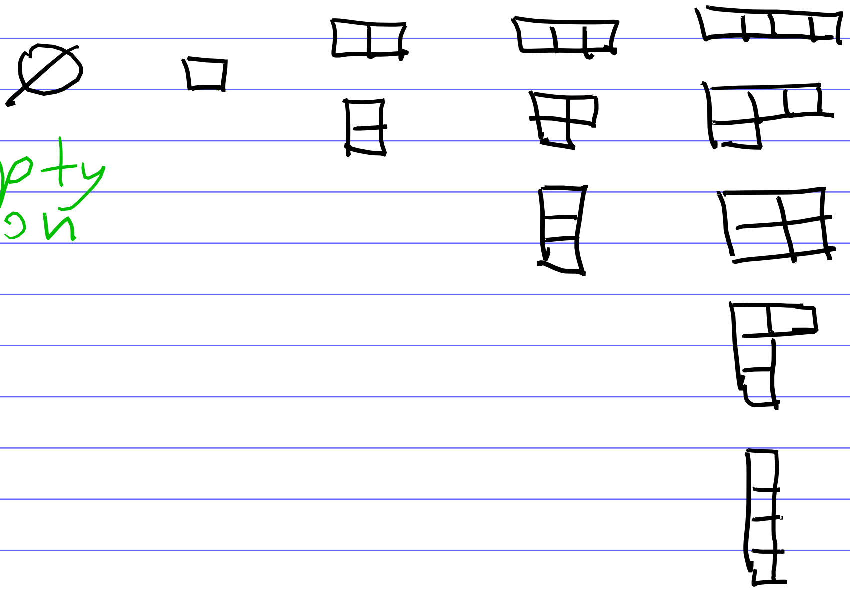
k	0	1	2	3	4	5
$p(k)$	1	1	2	3	5	7



the empty partition

Elementary symmetric functions:

$$e_k := m_{\underbrace{(1,1,\ldots,1)}_{k}} = \sum_{i_1 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

sum of all square-free monomials of degree k

Complete homogeneous symmetric functions

$$h_k := \sum_{i_1 \le i_2 \le \cdots \le i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

sum of all homogeneous degree k monomials

Power symmetric functions:

$$p_k := m_{(k)} = x_1^k + x_2^k + \cdots$$

For these 3 classes of sym. functions and a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ define

$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_\ell}$$

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}$$

$$p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}$$

For example,

$$e_{\tiny\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}} = e_{(1,1)} = e_1 \cdot e_1 = (x_1 + x_2 + \dots)^2$$

$$p_{\tiny\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}} = p_2 \cdot p_1 = (x_1^2 + x_2^2 + \dots) \cdot (x_1 + x_2 + \dots)$$

We've got 4 classes of symmetric functions m_λ, e_λ, h_λ, p_λ all labelled by partitions (or Young diagrams) λ.

Theorem

$\{m_\lambda\}$, $\{e_\lambda\}$, $\{h_\lambda\}$ are linear bases of Λ.

$\{p_\lambda\}$ is a $\mathbb{Q}$-linear basis of $\Lambda_{\mathbb{Q}} := \Lambda \otimes \mathbb{Q}$.

This is basically the Fundamental Theorem of Symmetric Functions.

Fundamental Theorem of Symmetric Functions (e-version)

Any symmetric function $f \in \Lambda$ can be uniquely written as a polynomial with integer coefficients in $e_1, e_2, e_3, \ldots$ i.e

$$\Lambda = \mathbb{Z}[e_1, e_2, \ldots].$$

Examples

$$h_2 = \sum_{i \le j} x_i x_j = m_{(2)} + m_{(1,1)}$$

$$= (x_1 + x_2 + \dots)^2 - \sum_{i<j} x_i x_j$$

$$= e_1^2 - e_2 = e_{(1,1)} - e_{(2)}.$$

$$m_{(2,1)} = x_1^2 x_2 + x_2^2 x_1 + \dots$$

$$= e_2 \cdot e_1 - 3 e_3.$$

Proof of FTSF (for e_k's)

Clearly, e_λ is a $\mathbb{Z}$-linear combination of m_μ's.
We need to show that each m_λ is a $\mathbb{Z}$-linear combination of e_μ's.

Fix degree k.

Let us show that the transition matrix between $\{e_\lambda \mid \lambda \vdash k\}$ and $\{m_\mu \mid \mu \vdash k\}$ is an upper triangular matrix with 1's on the main diagonal.

($\Longrightarrow$ this matrix is invertible over $\mathbb{Z}$ & we are done)

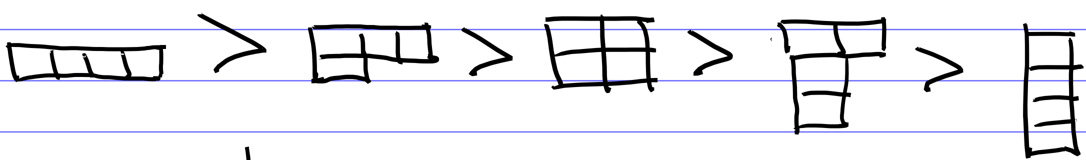
Need to order partitions λ

Order partitions lexicographically

$$\lambda >_{lex} \mu \quad \text{if}$$

$\lambda_1 > \mu_1$ or $(\lambda_1 = \mu_1 \ \& \ \lambda_2 > \mu_2)$, or $(\lambda_1 = \mu_1, \lambda_2 = \mu_2, \lambda_3 > \mu_3)$, etc.

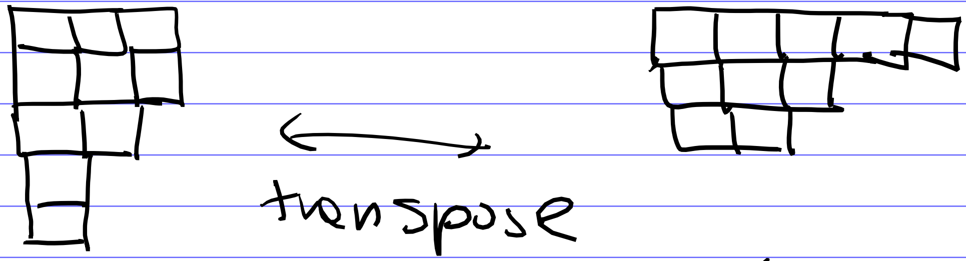
<u>Ex.</u> Partitions of 4.



lex. order

Def. The conjugate partition λ' to λ obtained by transposing the Young diagram

Ex $\lambda = (3,3,2,1,1)$



$\lambda = (3,3,2,1,1)$ $\qquad\qquad$ $\lambda' = (5,3,2)$

Lemma

some integer coeff.

$$e_\lambda = m_{\lambda'} + \sum_{\mu <_{lex} \lambda'} a_{\lambda\mu} m_\mu$$

Example. $e_{(4,1)} = e_4 \cdot e_1 =$

$(x_1x_2x_3x_4 + \dots)(x_1 + x_2 + \dots)$

$= \underline{x_1x_2x_3x_4 \cdot x_1} + \dots$

$= \underline{m_{(4,1)'}} + \dots$

the "leading monomial" function m_μ with lex. maximal μ.

In general, $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$

$= (x_1 \dots x_{\lambda_1} + \dots) \cdots (x_1 \dots x_{\lambda_\ell} + \dots)$

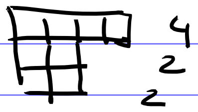
$= x_1 \dots x_{\lambda_1} \, x_1 \dots x_{\lambda_2} \cdots x_1 \dots x_{\lambda_\ell} + \dots$

$= x_1^{\lambda_1'} x_2^{\lambda_2'} \cdots x_r^{\lambda_r'} + \dots$

$= m_{\lambda'} + \begin{pmatrix} \text{lin. comb. of } m_\mu \\ \text{with } \mu <_{lex} \lambda' \end{pmatrix};$

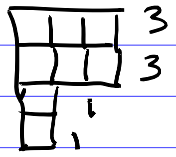
as needed.

Ex. $\lambda = (4,2,2)$

$\lambda =$ 

$(x_1x_2x_3x_4) \cdot (x_1x_2) \cdot (x_1x_2)$

$= x_1^3 x_2^3 x_3 x_4$

$\lambda' =$ 

Thus

$$\{e_\lambda \mid \lambda \vdash k\} = A \cdot \{m_\mu \mid \mu \vdash k\}$$

where A is an integer upper-triang. matrix with 1's on the diagonal (if we appropriately order λ's & μ's) of size $p(k) \times p(k)$.

$\Rightarrow$ A is invertible $/\mathbb{Z}$

$\Rightarrow$ Each m_λ is an integer lin. combination of e_μ's, as needed. $\square$

Exercise Prove the following version of FTSF for the homogeneous functions h_k's.

Theorem

$$\Lambda = \mathbb{Z}[h_1, h_2, h_3, \dots].$$