<u>Facts</u> about representations of a finite group G.

"Representations" = Finite dimensional linear representations over $\mathbb{C}$.

A <u>representation</u> of G is the same thing as <u>G-module</u>.

– a vector space $V \simeq \mathbb{C}^d$ with a group action: $\forall g \in G$, $v \in V$ we have $g \cdot v$.

An <u>irreducible</u> representation is a representation V that does not have nontrivial subrepresent., i.e. proper G-invariant subspaces $W \subset V$.

- <u>Maschke's Theorem</u> $\Rightarrow$ Every represent. decomposes into a direct sum of irreducible representations.

- G has finitely many equivalence classes of irreducible repsentations $V_1, V_2, \ldots, V_N$

N = # conjugacy classes in G

$(g_1 \sim g_2$ if $\exists\, c \in G$ s.t. $g_2 = c g_1 c^{-1})$

We'll denote $G^\wedge := \{V_1, \ldots, V_N\}$

For $G = S_n$: 2 permutation in S_n are conjugate if they have the same cyclic type

$(1\,5\,2\,3)(6\,8)(4)(7) \sim$
$(7\,2\,8\,6)(1\,4)(3)(5)$

Conjugacy classes of $S_n \overset{bij}{\longleftrightarrow}$ partitions of n.

$\# G^\wedge = \#$ Irreps. of $S_n = p(n)$
(# partitions of n).

<u>Example</u> S_2 has 2 irreducible reps.: $\mathbb{1}$ (trivial rep), sign rep.

S_3 has 3 irreps: 1, sign, and 2-dimensional subrepresent. of the defining representation:
$V = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$ S_3 acts by permutations of x_1, x_2, x_3.

By Maschke's Thm, for any rep. V

$$V = \underbrace{V_1 \oplus \dots \oplus V_1}_{m_1} \oplus \underbrace{V_2 \oplus \dots \oplus V_2}_{m_2} \oplus \cdots \oplus \underbrace{V_N \oplus \dots \oplus V_N}_{m_N}$$

<u>Schur's lemma</u> $\Rightarrow$

- The multiplicities $m_1, \dots, m_N$ are uniquely defined by V (i.e. any other decomp. of V into irreps has the same multiplicities).

Moreover, if all $m_i \in \{0, 1\}$ then there is a <u>unique</u> decomp. of V into irreducible reps.

Group algebre $\mathbb{C}[G] =$

$$= \Big\{ \sum_{g\in G} \alpha_g \, g \Big\} \qquad \alpha_g \in \mathbb{C}$$

formal linear combinations of elements g of G

(as a linear space $\mathbb{C}[G] \simeq \mathbb{C}^{|G|}$)

Theorem. We have the fundamental isomorphism:

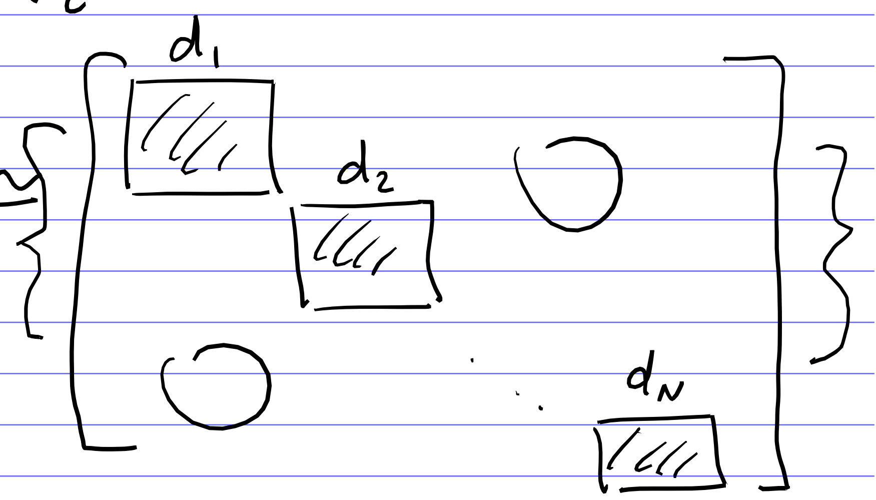
$$\boxed{\mathbb{C}[G] \simeq \bigoplus_{i=1}^{N} \mathrm{End}(V_i)}$$

($\mathrm{End}(V_i)$ algebra of endomorphisms of V_i, i.e. linear maps $V_i \to V_i$)

More concretely, if we pick a linear basis in each V_i, then $\mathrm{End}(V_i) \simeq \Big\{ \begin{array}{c} d_i \times d_i \\ \text{matrices} \end{array} \Big\}$

$d_i = \dim V_i$

$\mathbb{C}[G] \simeq$



the algebra of block-diagonal matrices with square blocks of sizes $d_i = \dim V_i$.

So $\dim \mathbb{C}[G] = |G|$

$$= \sum_i (d_i)^2.$$

__Example__ $G = S_2$

$$\mathbb{C}[S_2] = \{a\mathbb{1} + b\, s_1\}$$

$$= \left\{c\left(\frac{1+s_1}{2}\right) + d\left(\frac{1-s_1}{2}\right)\right\}$$

$\{1, s_1\}$
an obvious basis of $\mathbb{C}[S_2]$

$\left\{\frac{1+s_1}{2}, \frac{1-s_1}{2}\right\}$
a "better" basis

trivial rep

$$\simeq \left\{\begin{bmatrix} \boxed{c} & 0 \\ 0 & \boxed{d} \end{bmatrix}\right\}$$

sign rep.

Multiplication of elements in $\mathbb{C}[S_2]$ correspond to mult. of matrices:

$$\left(c\left(\frac{1+s_1}{2}\right) + d\left(\frac{1-s_1}{2}\right)\right)\cdot\left(c'\left(\frac{1+s_1}{2}\right) + d'\left(\frac{1-s_1}{2}\right)\right)$$

$$= cc'\left(\frac{1+s_1}{2}\right) + dd'\left(\frac{1-s_1}{2}\right).$$

$$\frac{1+s_1}{2}\cdot\frac{1+s_1}{2} = \frac{1+s_1}{2}$$

$$\frac{1-s_1}{2}\;\frac{1-s_1}{2} = \frac{1-s_1}{2}$$

$$\frac{1+s_1}{2}\cdot\frac{1-s_1}{2} = \frac{1-s_1}{2}\cdot\frac{1+s_1}{2} = 0$$

For S_3

$$\mathbb{C}[S_3] \simeq \left(\left[\begin{array}{c|cc|c} \boxed{a} & 0 & 0 & 0 \\ \hline 0 & b & c & 0 \\ 0 & d & e & 0 \\ \hline 0 & 0 & 0 & \boxed{f} \end{array} \right] \right)$$

trivial rep

2 dim irrep (Sub rep. of the defining rep.)

sign rep.

Def For an algebra $\mathcal{A}$, the <u>center</u> of $\mathcal{A}$ is

$$Z_{\mathcal{A}} := \{ c \in \mathcal{A} \mid c \cdot a = ac \ \forall a \in \mathcal{A} \}$$

Example The center of the algebra of $n \times n$ matrices is

$$Z(\mathrm{Mat}(n,n)) = \left\{ \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} \right\}$$

Let's describe the center of $\mathbb{C}[G]$

1st description

$$\sum_{g \in G} \alpha_g g \in Z_{\mathbb{C}[G]}$$

$$\Updownarrow$$

$$g' \left(\sum \alpha_g g \right) = \left(\sum \alpha_g g \right) g' \quad \forall g' \in G$$

$$\Updownarrow$$

$$\alpha_{gg'} = \alpha_{g'g} \quad \forall g, g' \in G$$

$$\Updownarrow$$

$\alpha : g \mapsto \alpha_g$ is a <u>class function</u> on G

Def A <u>class function</u> $G \to \mathbb{C}$ is a fuction constant on conjugacy classes of G.

In particular, $\dim Z_{\mathbb{C}[G]}$ $= \#$ conjugacy classes in G.

2^{nd} description of the center $Z_{\mathbb{C}[G]}$.

$$\mathbb{C}[G] \simeq \left\{ \begin{bmatrix} \boxed{}^{\,d_1} & & & \\ & \boxed{}^{\,d_2} & & \\ & & \ddots & \\ & & & \boxed{}^{\,d_N} \end{bmatrix} \right\}$$

$Z_{\mathbb{C}[G]} \simeq$ the subalgebra of matrices

$$\begin{bmatrix} \boxed{\begin{matrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_1 \end{matrix}} & & & \text{\Large 0} \\ & \boxed{\begin{matrix} \lambda_2 & & 0 \\ & \ddots & \\ 0 & & \lambda_2 \end{matrix}} & & \\ & & \ddots & \\ \text{\Large 0} & & & \boxed{\begin{matrix} \lambda_N & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{matrix}} \end{bmatrix}$$

In particular,

$\dim Z_{\mathbb{C}[G]} = \#$ of equivalence classes of irreducible represent. of G.

Vershik-Okounkov's "new approach" to the representation theory of S_n.

Key idea: We should study the chain of included symmetric groups: $S_0 \subset S_1 \subset S_2 \subset S_3 \subset \dots$

(rather than study one particular symmetric group S_n).

The standard inclusion of symmetric groups:

$$S_{n-1} \hookrightarrow S_n$$

$$\begin{array}{c} \cup \\ w = \begin{pmatrix} 1 & 2 & \dots & n-1 \\ w_1 & w_2 & \dots & w_{n-1} \end{pmatrix} \end{array} \longmapsto \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ w_1 & w_2 & \dots & w_{n-1} & n \end{pmatrix}$$

Let's first do this in a more general setting:

$$G_0 = \{1\} \subset G_1 \subset G_2 \subset \dots$$

any family of included (finite) groups G_n.

$\hat{G_n}$ = set of equiv. classes of irreducible reps. of G_n

We'll denote elements of $\hat{G_n}$ by λ, and corresponding irreducible reps. by V_λ.

(In case of symmetric groups, λ's will be Young diagrams. But let us now treat λ's just as indices of V_λ's.)

Bratteli diagram of $G_0 \subset G_1 \subset G_2 \subset \dots$

is the directed graph:

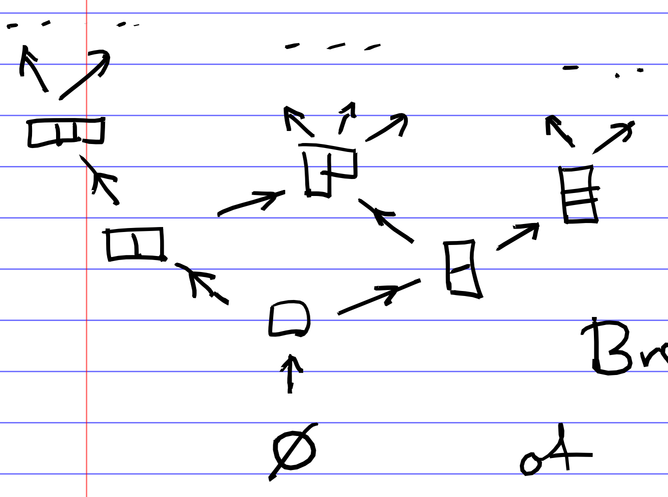
- set of vertices $= \hat{G_0} \cup \hat{G_1} \cup \hat{G_2} \cup \dots$
- For $\lambda \in \hat{G_n}$, $\mu \in \hat{G_{n-1}}$ we have a directed edge $\mu \xrightarrow{m} \lambda$ of multiplicity m

if V_μ appears with multiplicity m in the restriction of V_λ to G_{n-1}.

$$\mathrm{Res}^{G_n}_{G_{n-1}} V_\lambda = \dots \underbrace{\oplus V_\mu \oplus \dots \oplus V_\mu \oplus}_{m \text{ times}} \dots$$

The Bratteli diagram has a unique source $\emptyset \in \hat{G_0}$.

(For $S_0 \subset S_1 \subset S_2 \subset \dots$ the Bratteli diagram is the Young's lattice $\mathbb{Y}$. But let us pretend that we don't know this yet.)



Bratteli diagram of $S_0 \subset S_1 \subset S_2 \subset \dots$

Theorem

$\dim V_\lambda$ = # directed paths $\varnothing \to \cdots \to \lambda$ in the Bratteli diagram.

Proof $\dim V_\lambda =$

$$= \dim\left(\mathrm{Res}^{G_n}_{G_{n-1}} V_\lambda\right)$$

$$= \sum_{\mu \in \hat{G}_{n-1}} (\text{mult. of edge } \mu \to \lambda) \dim V_\mu .$$

$$= \ldots = \#\,\text{paths}(\varnothing \to \cdots \to \lambda).$$

$(\dim V_\varnothing = 1)$

We'll denote such chains by

$$T = (\varnothing \to \cdots \to \lambda).$$

(For symm. groups, they corresp. to SYT's. But we don't know this yet.)

We want to construct a linear basis $\{v_T\}$ in V_λ labelled by chains $T = (\emptyset \to \ldots \to \lambda)$.

<u>Gelfand-Tsetlin basis</u> of V_λ

$\lambda \in \hat{G_n}$,

$\mathrm{Res}^{G_n}_{G_{n-1}} V_\lambda = \bigoplus V_\mu \qquad \mu \in \hat{G}_{n-1}$

Then restrict each V_μ to G_{n-2}:

$\mathrm{Res}^{G_{n-1}}_{G_{n-2}} V_\mu = \bigoplus V_\nu \quad , \nu \in \hat{G}_{n-2}$

etc.

In the end, each V_λ decomposes into a direct sum of irreps of $G_0 = \{1\}$ (which are 1-dim spaces), where the summands corresp. to all paths $T = (\emptyset \to \ldots \to \lambda)$

Pick a unit vector v_T in each 1-dim'l component of V_λ. We obtain a basis $\{v_T\}$ of V_λ.

Assume that the Bratteli diagram is <u>multiplicity-free</u> (no multiple edges), then at each stage the decomp.

$$\mathrm{Res}^{G_n}_{G_{n-1}} V_\lambda = \bigoplus V_\mu = \dots$$

is <u>unique</u>

$\Rightarrow$ The Gelfand-Tsetlin basis is unique up to rescaling the vectors v_T.

<u>Example</u> $S_0 \subset S_1 \subset S_2 \subset S_3 \subset \dots$

$$V_\lambda = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$$

2-dimensional irrep. of S_3

trivial rep. of S_2

$$\mathrm{Res}^{S_3}_{S_2} V_\lambda = \langle (1,1,-2) \rangle \oplus \langle (1,-1,0) \rangle$$

sign rep. of S_2

$$\mathrm{Res}^{S_2}_{S_1}, \ \mathrm{Res}^{S_1}_{S_0} \rightsquigarrow \langle (1,1,-2) \rangle \oplus \langle (1,-1,0) \rangle$$

Gelfand-Tsetlin basis V_λ:

$$v_1 = \frac{1}{\sqrt{6}}(1,1,-2), \quad v_2 = \frac{1}{\sqrt{2}}(1,-1,0)$$

Action of S_3 on the GT-basis

$$s_1 : \quad v_1 \longmapsto v_1 \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$v_2 \longmapsto -v_2$$

$$s_2 : \quad v_1 \mapsto \frac{1}{\sqrt{6}}(1,-2,1) = -\frac{1}{2} v_1 + \frac{\sqrt{3}}{2} v_2$$

$$v_2 \mapsto \frac{1}{\sqrt{2}}(1,0,-1) = \frac{\sqrt{3}}{2} v_1 + \frac{1}{2} v_2$$

$$s_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

(Check that $s_1 s_2 s_1 = s_2 s_1 s_2$ for these matrices)

Let $\mathbb{C}[G_0] \subset \mathbb{C}[G_1] \subset \mathbb{C}[G_2] \subset \dots$ the chain of included group algebras $\mathbb{C}[G_n]$.

Let $Z_n := Z_{\mathbb{C}[G_n]}$ the center of the group algebra $\mathbb{C}[G_n]$.

Def. The Gelfand-Tsetlin subalgebra $GT_n \subset \mathbb{C}[G_n]$ is the subalgebra of $\mathbb{C}[G_n]$ generated by $Z_1, Z_2, \dots, Z_n$.

Recall $\mathbb{C}[G_n] \simeq \left\{ \begin{bmatrix} \boxed{\ } & & 0 \\ & \boxed{\ } & \\ 0 & & \ddots \ \boxed{\ } \end{bmatrix} \right\}$

blocks corresp. to elements of $\hat{G}_n$.

matrices expressed in the Gelfand-Tsetlin bases for all V_λ $\lambda \in \hat{G}_n$.

Proposition GZ_n is the algebra of all diagonal matrices (w.r.t. Gelfand-Tsetlin bases).

It is a maximal commutative subalgebra in $\mathbb{C}[G_n]$.

Ex. $\mathbb{C}[S_3] \simeq \left\{ \begin{bmatrix} \boxed{a} & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ 0 & 0 & 0 & \boxed{f} \end{bmatrix} \right\}$

$GT(S_3) \simeq \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & f \end{bmatrix} \right\}$

Remark. In Vershik-Okounkov's theory GT algebra plays a role similar to Cartan subalgebra in Lie theory.

In order to describe irreps. V_λ of G_n, we need to understand the eigenvalues of elements of GT_n

Let us now assume that $G_n = S_n$ with standard inclusions

$$S_0 \subset S_1 \subset S_2 \subset \dots$$

$$Z_n = Z\,\mathbb{C}[S_n] = \left\{ \sum_{w \in S_n} \alpha_w w \right\}$$

α_w is a class function, i.e. α_w depends only on the cyclic type of permutation w.

GT_n = alg. generated by $Z_1, \dots, Z_n$

Young-Jucys-Murphy elements

$X_1, X_2, X_3, \dots$

$(X_n \in \mathbb{C}[S_n])$

$X_1 = 0$

$X_2 = (1,2)$ transposition of 1 & 2

$X_3 = (1,3) + (2,3)$ sum of two transpositions

...

$X_i = (1,i) + (2,i) + \dots + (i-1,i)$

a class function in S_n

Clearly, $X_n = \sum(\text{all transp. in } S_n)$

$- \sum(\text{all transp. in } S_{n-1})$

class funct in S_{n-1}

So $X_n \in \langle Z_n, Z_{n-1} \rangle$

center of $\mathbb{C}[S_n]$ center of $\mathbb{C}[S_n]$

So $X_1, \dots, X_n \in GT_n$

"Crucial" Theorem

$$GT_n = \langle X_1, \dots, X_n \rangle$$

the subalgebra of $\mathbb{C}[S_n]$ generated by YJM-elements

---

Idea of proof. Need to show that any class function in S_n, say, any

$$C_\lambda = \sum_{\substack{w \in S_n \\ \text{cycle type } \lambda}} w$$

can be expressed in terms of $X_1, \dots, X_n$. $\square$

For example,

$$\sum\begin{pmatrix}\text{all transpositions} \\ \text{in } S_n\end{pmatrix} = X_1 + \dots + X_n.$$

Exercise Express

$\sum (a, b, c)$ in terms of $X_1, X_2, \dots, X_n$.

all 3-cycles in S_n

---

e.g. $X_2 \cdot X_3 = ((1,2))((1,3)+(2,3))$

$= (1,2,3) + (1,3,2).$

X_i's act diagonally on GT basis $\{v_T\}$ for each irrep, V_λ

In order to describe irreps. we need to analyze the eigenvalues of X_i

Basis elts v_T of V_λ

$\updownarrow$

collections of vectors $a_T = (a_1, \dots, a_n)$ (a_i is the eigenvalue of X_i)

$$\underline{X_i \cdot v_T = a_{T,i} \, v_T}$$

We'll see that this leads directly to combinatorics of Young diagrams, Young tableaux, etc.

Theorem Elements $s_1, \ldots, s_{n-1}$ and $X_1, \ldots, X_n$ in $\mathbb{C}[S_n]$ satisfy the relations:

- $X_i X_j = X_j X_i \quad \forall i,j$
- $s_i X_j = X_j s_i, \quad j \neq i, i+1$
- $s_i X_i + 1 = X_{i+1} s_i$
- $s_i X_{i+1} - 1 = X_i s_i$

This is called <u>degenerate Hecke algebra</u> (DHA).

Proof Easy direct verification. $\square$

We'll see DHA relations

$\rightsquigarrow$ combinatorial description of eigenvalues of X_i's

$\rightsquigarrow$ Young tableaux.