18.217 Lecture 14 10/05/2020

Last time: Jacobi-Trudi identities:

$$s_{\lambda/\mu} = \det\left(h_{\lambda_i - i - \mu_j + j}\right)_{i,j=1}^{n}$$

$$s_{\lambda/\mu} = \det\left(e_{\lambda'_i - i - \mu'_j + j}\right)_{i,j=1}^{m}$$

$n \geq$ # rows of λ/μ

$m \geq$ # columns of λ/μ.

By convention, $h_0 = e_0 = 1$ and $h_k = e_k = 0$ for $k<0$.

We've proved J.-T. identities for (combinatorially defined) $s_{\lambda/\mu}$ using Gessel-Viennot method (Lindström's Lemma)

Applications of J.-T. identities:

(1) the involution ω: $s_k \overset{\omega}{\longleftrightarrow} h_k$

$$\omega(s_{\lambda/\mu}) = s_{(\lambda/\mu)'}$$

(2) combinatorial def. of $s_\lambda \Longleftrightarrow$ classical def. of s_λ

(We already mentioned several appoaches to prove this. Let us now give a complete proof.)

Theorem For combinatorially defined s_λ, we have
$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\delta}}{a_\delta},$$
where $a_\alpha := \det\left([x_j^{\alpha_i}]_{i,j=1}^{n} \right)$,

for $\alpha = (\alpha_1, \dots, \alpha_n)$,

$\delta = (n-1, n-2, \dots, 1, 0)$.

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$

Define the $n \times n$ matrices

$$A_\alpha := \left(x_j^{\alpha_i} \right)_{i,j=n}^{n}. \quad \left(\text{So } a_\alpha = \det(A_\alpha) \right)$$

$$H_\alpha := \left(h_{\alpha_i - n + j}(x_1, \dots, x_n) \right)_{i,j=1}^{n}$$

$$E := \left((-1)^{n-i} e_{n-i}^{(j)} \right)_{i,j=1}^{n},$$

where $e_{n-i}^{(j)} := e_{n-i}(x_1, \dots x_{j-1}, \widehat{x_j}, x_{j+1}, \dots, x_n)$

skip this variable

Lemma $A_\alpha = H_\alpha \cdot E$

Proof. Use generating functions:

$$E^{(j)}(t) := \sum_{k \geq 0} e_k^{(j)} t^k =$$

$$= (1+x_1 t) \dots (1+x_{j-1} t)(1+x_{j+1} t) \dots (1+x_n t).$$

$$= \prod_{i \in [n] \setminus \{j\}} (1+x_i t).$$

$$H(t) := \sum_{k \geq 0} h_k(x_1 \dots x_n)\, t =$$

$$= \frac{1}{1-x_1 t} \cdot \dots \cdot \frac{1}{1-x_n t} = \prod_{i=1}^{n} \frac{1}{1-x_i t}.$$

$$H(t)\, E^{(j)}(-t) = \frac{1}{1-x_j t}$$

Take the coeff. of t^{α_i} on both sides.

$$\sum_{\substack{a+b=\alpha_i \\ a,b \geq 0}} h_a (-1)^b e_b^{(j)} = \underbrace{x_j^{\alpha_i}}_{(A_\alpha)_{ij}}$$

$b = n-k$
$a = \alpha_i - b$

$$\shortparallel$$

$$\sum_{k=0,\dots,n} \underbrace{h_{\alpha_i - n + k}}_{(H_\alpha)_{ik}} \cdot \underbrace{(-1)^{n-k} e_{n-k}^{(j)}}_{(E)_{kj}}$$

So $H_\alpha \cdot E = A_\alpha$,

as needed. $\square$

Proof of Theorem $\left(s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}\right)$

We have $a_\alpha = |A_\alpha| = |H_\alpha| \cdot |E|$.

For $\alpha = \delta$: $a_\delta = \begin{vmatrix} 1 & h_1 & h_2 & \cdots \\ & 1 & h_1 & h_2 \\ & & \ddots & \ddots \\ 0 & & & 1 \end{vmatrix} \cdot |E|$

H_δ is an upper-triang. matrix with 1's on diag.

So $|E| = a_\delta$

Take $\alpha = \lambda + \delta$:

$$a_{\lambda+\delta} = |A_{\lambda+\delta}| = |H_{\lambda+\delta}| \cdot |E|$$

$$= |H_{\lambda+\delta}| \cdot a_\delta .$$

Jacobi-Trudi det. for $s_\lambda(x_1 \dots x_n)$

So $s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\delta}}{a_\delta}$. $\square$

Application #3 of J-T.

Determinantal formula for #SYT's

Let $f_{\lambda/\mu} :=$ SYT's of shape λ/μ

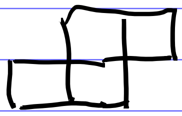
<u>Theorem</u>. $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_n)$ partitions s.t. $\lambda \supseteq \mu$.

$N = |\lambda/\mu| \leftarrow$ # boxes in λ/μ

$$f_{\lambda/\mu} = N! \det\left(\left[\frac{1}{(\lambda_i - i - \mu_j + j)!}\right]_{i,j=1}^{n}\right)$$

(Conventions: $0! = 1$, $\frac{1}{k!} = 0$, $k < 0$)

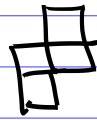
that is $k! = \infty$

<u>Example</u>. $\lambda/\mu =$ 

$\lambda = (3,2)$, $\mu = (1,0)$, $n = 2$, $N = 4$

f  $= 4! \begin{vmatrix} \frac{1}{2!} & \frac{1}{4!} \\ \frac{1}{0!} & \frac{1}{2!} \end{vmatrix} =$

$= 4! \left(\frac{1}{2} \cdot \frac{1}{2} - \frac{1}{4!}\right) = 5.$

Also $(\lambda/\mu)' =$  $\Leftarrow$ the same # of SYT's.

$\lambda' = (2,2,1)$ $\quad \mu' = (1,0,0)$, $n=3$, $N=4$

f 

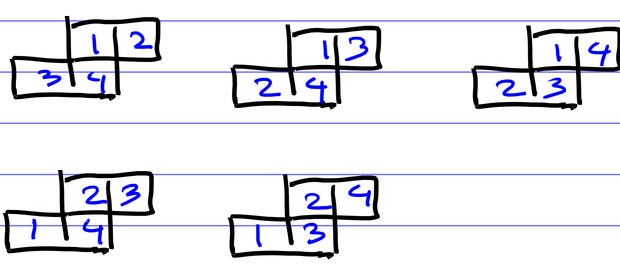
$$= 4! \begin{vmatrix} \frac{1}{1!} & \frac{1}{3!} & \frac{1}{4!} \\ \frac{1}{0!} & \frac{1}{2!} & \frac{1}{3!} \\ \frac{1}{(-2)!} & \frac{1}{0!} & \frac{1}{1!} \end{vmatrix}$$

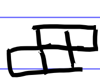
$$= 4! \begin{vmatrix} 1 & \frac{1}{3!} & \frac{1}{4!} \\ 1 & \frac{1}{2!} & \frac{1}{3!} \\ 0 & 1 & 1 \end{vmatrix} =$$

$$= 4!\left(1 \cdot \frac{1}{2} \cdot 1 - \frac{1}{3!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$$= 12 - 4 - 4 + 1 = 5.$$

SYT's:



f  $= 5.$

Compare

$$s_{\lambda/\mu} = \det\left(\left(h_{\lambda_i - i + \mu_j - j}\right)_{i,j=1}^{n}\right)$$

$$f_{\lambda/\mu} = N!\,\det\left(\left(\frac{1}{(\lambda_i - i + \mu_j - j)!}\right)_{i,j=1}^{n}\right)$$

$$s_{\lambda/\mu} \overset{?}{\longleftrightarrow} f_{\lambda/\mu}$$

count SSYT's   count SYT's

$f_{\lambda/\mu}$ = the coeff. of $x_1 \dots x_N$ in $s_{\lambda/\mu}$, where $N = |\lambda/\mu|$.

Notation: $[x_1 \dots x_N]\,(s_{\lambda/\mu})$

the coeff. of $x_1 \dots x_N$ in $s_{\lambda/\mu}$

Recall, the p-version of the fundamental theorem of sym. funct.

$\Lambda_{\mathbb{Q}} := \mathbb{Q} \otimes \Lambda$ the ring of symm. funct. with rational coefficients

$$\Lambda_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, p_3, \dots]$$

every symmetric function can be uniquely expressed as a polynomial in p_k's with rational coefficients.

$$p_k := x_1^k + x_2^k + x_3^k + \dots$$

$$k \geq 1$$

Exponential specialization of symmetric functions

$$ex : \Lambda_{\mathbb{Q}} \longrightarrow \mathbb{Q}[t]$$

$$g(p_1, p_2, \dots) \longmapsto g(t, 0, 0, \dots)$$

ex is a homomorphism of rings.

Examples: (1) $e_1 = p_1$.

So $ex(e_1) = ex(p_1) = t$

(2) $e_2 = \frac{1}{2}(p_1^2 - p_2)$.

So $ex(e_2) = ex\left(\frac{1}{2}(p_1^2 - p_2)\right)$

$$= \frac{t^2}{2}.$$

(3) $h_2 = \frac{1}{2}(p_1^2 + p_2)$

So $ex(h_2) = \frac{1}{2}t^2$.

Lemma. $f(x_1, x_2, \dots) \in \Lambda_{\mathbb{Q}}$

Then $ex(f) = \sum_{N \geq 0} [x_1 \dots x_N](f) \frac{t^N}{N!}$

Proof. Enough to prove for

$f = p_\lambda := p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_k}$ ← form a linear basis of $\Lambda_{\mathbb{Q}}$.

$$ex(p_\lambda) := \begin{cases} t^N & \text{if } \lambda = (\underbrace{1 \dots 1}_{N}) \\ 0 & \text{otherwise} \end{cases}$$

$[x_1 \dots x_N](p_\lambda) =$

$[x_1 \dots x_N]\left((x_1^{\lambda_1} + x_2^{\lambda_1} + \dots) \cdot \right.$

$\left. \cdot (x_1^{\lambda_2} + x_2^{\lambda_2} + \dots) \cdot (x_1^{\lambda_3} + x_2^{\lambda_3} + \dots) \dots \right)$

$$= \begin{cases} N! & \text{if } \lambda = (\underbrace{1 \dots 1}_{N}) \\ 0 & \text{otherwise} \end{cases}$$

So Lemma follows. $\square$

Example $ex(h_k) = \frac{t^k}{k!}$ if $k \geq 0$

(or $= 0$ if $k < 0$).

Now apply the exponential specialization to Jacobi-Trudi

$$S_{\lambda/\mu} = \det\left(\left(h_{\lambda_i - i - \mu_j + j}\right)\right)$$

$$\Big\downarrow \text{ex}$$

$$\frac{f_{\lambda/\mu}}{N!} t^N = \det\left(\left[\frac{t^{\lambda_i - i - \mu_j + j}}{(\lambda_i - i - \mu_j + j)!}\right]\right)$$

$N = |\lambda/\mu|$

$$= t^N \cdot \det\left(\left[\frac{1}{(\lambda_i - i - \mu_j + j)!}\right]\right)$$

$\square$

In particular,

$$f_\lambda = N! \left(\left[\frac{1}{(\lambda_i - i + j)!}\right]_{i,j=1}^{n}\right)$$

$N = |\lambda|$.

<u>Remark.</u>

We already proved the hook-length formula for f_λ using RSK or Hillman-Gressl.

One can also deduce it from the determinantal formula

$$\det\left(\left(\frac{1}{\lambda_i - i + j}\right)_{i,j=1}^{n}\right) \overset{\text{Some matrix manipulations}}{=}$$

$$= \prod_{a \in \lambda} \frac{1}{h_a}.$$

3 principal specializations of symmetric functions $f \in \Lambda_{\mathbb{Q}}$

(1) $f \longmapsto f(1, q, q^2, \dots, q^{n-1}) \in \mathbb{Q}[q]$

(2). $f \longmapsto f(\underbrace{1, 1, \dots, 1}_{n}) \in \mathbb{Q}$

(3) $f \longmapsto f(1, q, q^2, \dots) \in \mathbb{Q}[[t]]$.

this is called the stable principal specialization.

Clearly, (1) is the most general among these 3 specializ.
(2) is lim. of (1) as $q \to 1$
(3) is lim. of (1) as $n \to \infty$.

q-analogs

a "classical" object $A \rightsquigarrow$ "quantum" object $A(q)$ s.t. $A(q) \to A$ as $q \to 1$ (the classical limit)

q-numbers:

$$[k]_q = 1+q+q^2+\dots+q^{k-1} = \frac{1-q^k}{1-q}$$

$k \in \mathbb{Z}_{\geq 0}$

q-factorial

$$[n]_q! = [1]_q [2]_q \cdot \ldots \cdot [n]_q = \prod_{k=1}^{n} \frac{1-q^k}{1-q}$$

q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! \, [n-k]_q!}$$

$$= \frac{(1-q)(1-q^2)\dots(1-q^n)}{(1-q)\cdots(1-q^k)\cdot(1-q)\cdots(1-q^{n-k})}.$$

Theorem $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q with positive integer coeffs.

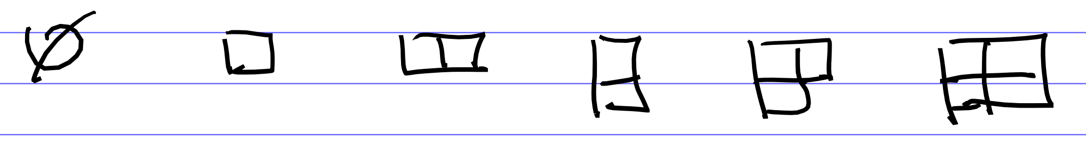
Explicitly

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\lambda \subseteq k \times (n-k)} q^{|\lambda|}$$

the sum over Young diagrams λ that fit inside the $k \times (n-k)$ rectangle.

Example. $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{\cancel{(1-q)}\cancel{(1-q^2)}(1-q^3)(1-q^4)}{\cancel{(1-q)}\cancel{(1-q^2)}(1-q)(1-q^2)}$

$= (1+q+q^2)(1+q^2) = 1+q+2q^2+q^3+q^4$



$= 1 + q + q^2 + q^2 + q^3 + q^4.$

Proof (a boring proof) Check that both sides satisfy the same recurrence relations.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q . \square$$

q-Pascal's triangle:

1

1 $\searrow^{1}$ $\swarrow_{q}$ 1

1 $\searrow^{1}$ $1+q$ $\swarrow_{q}$ $\searrow^{1}$ $\swarrow_{q^2}$ 1

1 $\searrow^{1}$ $\swarrow_{q}$ $[3]_q$ $\searrow^{1}$ $\swarrow_{q^2}$ $[3]_q$ $\searrow^{1}$ $\swarrow_{q^3}$ 1

1 $[4]_q$ $1+q+2q^2+q^3+q^4$ $[4]_q$ 1

. . . .

... back to principal specializations

(1) $f \mapsto f(1, q, q^2, \dots q^{n-1})$

<u>Lemma</u> $e_k \overset{p.s.}{\longmapsto} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$

$$h_k \overset{p.s.}{\longmapsto} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q$$

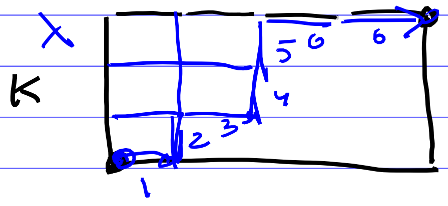
<u>Proof</u> $e_k(x_1 \dots x_n) =$

$= \sum$ square-free monomials in $x_1, \dots, x_n$

$\updownarrow$ bij.

Young diagrams $\lambda \subseteq k \times (n-k)$.

<u>Ex.</u> $n = 7$ $\quad x_2 x_4 x_5$



i - a vertical step for $x_i \in$ monomial

j - a horizontal step for $x_j \notin$ monomial

$$x_{i_1} \dots x_{i_k} \overset{p.s.}{\longmapsto} q^{i_1 - 1} q^{i_2 - 1} \dots q^{i_k - 1}$$

$$\| $$

$$q^{\binom{k}{2} + |\lambda|}$$

So $e_k \overset{p.s.}{\longmapsto} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$

A similar construction for h_k

<u>Corollary</u>. For $f \mapsto f(\underbrace{1,\dots,1}_{n},0,0,\dots)$

$$e_k \longmapsto \binom{n}{k} \qquad (\text{set } q=1)$$

$$h_k \longmapsto \binom{n+k-1}{k}$$

For $f \mapsto f(1,q,q^2,\dots)$

$$e_k \longmapsto \frac{q^{\binom{n}{2}}}{(1-q)(1-q^2)\cdots(1-q^k)}$$

$$h_k \longmapsto \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}$$

(take lim $n \to \infty$).

Apply these specializations to Jacobi-Trudi identities...

Corollary

$$s_{\lambda/\mu}(1, q, \ldots, q^{n-1}) =$$

$$= \det\left(\left(\begin{bmatrix} n + \lambda_i - i - \mu_j + j - 1 \\ \lambda_i - i - \mu_j + j \end{bmatrix}_q\right)_{i,j=1}^{n}\right)$$

q-binom coeffs.

$$s_{\lambda/\mu}(\underbrace{1 \ldots 1}_{n}) = \#\text{ SSYT of shape } \lambda/\mu \text{ w/ entries} \in \{1, \ldots, n\}$$

$$= \det\left(\begin{matrix}\text{some expression with} \\ \text{usual binomial coeffs}\end{matrix}\right)$$

$$s_{\lambda/\mu}(1, q, q^2, \ldots) =$$

rows in λ/μ.

$$= (1-q)^{-|\lambda/\mu|} \cdot \det\left(\left(\frac{1}{[\lambda_i - i - \mu_j + j]_q!}\right)_{i,j=1}^{n}\right)$$

In particular, for straight shapes.

<u>Compare</u>

det. formula

hook length formula

$$f_\lambda = N! \cdot \det\left(\left(\frac{1}{(\lambda_i - i + j)!}\right)\right) = \frac{N!}{\prod_{a \in \lambda} h_a}$$

$n \times n$ matrix

$$N = |\lambda|$$

$$s_\lambda(1, q, q^2, \ldots) = (1-q)^{-N} \det\left(\left(\frac{1}{[\lambda_i - i + j]_q!}\right)\right)$$

$$= q^{n(\lambda)} \prod_{a \in \lambda} \frac{1}{(1 - q^{h_a})}$$

Stanley's formula

$$= q^{n(\lambda)} (1-q)^{-N} \prod_{a \in \lambda} \frac{1}{[h_a]_q}$$

$$\left(n(\lambda) := \sum_{i \geq 0} (i-1) \lambda_i \right).$$

Stanley's formula is basically a q-analog of hook length formula.

More on q-binomial coeff.

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = c_0 + c_1 q + c_2 q^2 + \dots + c_M q^M$$

$$M = k(n-k)$$

c_i's are called the

<u>Gaussian coefficients</u>

$c_i :=$ # Young diagrams λ s.t.

- $\lambda \subseteq k \times (n-k)$
- $|\lambda| = i$

<u>Theorem</u>.

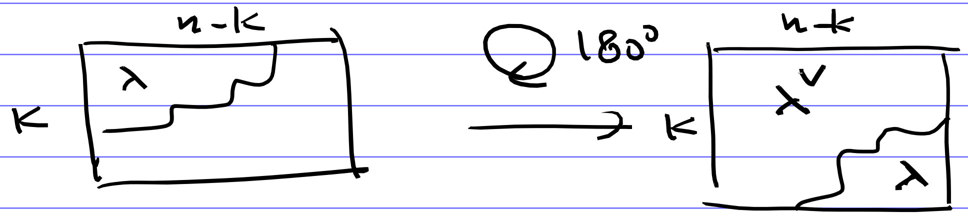
(1) $c_i = c_{M-i}$ (symmetry)

(2) $c_0 \le c_1 \le \dots \le c_{\lfloor \frac{M}{2} \rfloor} \ge \dots \ge c_M$

(unimodality)

$k \times (n-k) / \lambda$ rotated by $180°$

(1) is easy $\lambda \Longleftrightarrow \lambda^\vee$



$\lambda = (\lambda_1 \dots \lambda_k)$ $\quad \lambda^\vee = (n-k-\lambda_k, \dots, n-k-\lambda_1)$

$$|\lambda| = i \iff |\lambda^\vee| = k(n-k) - i.$$

(2) more complicated...

Sylvester's proof of the unimodality of Gaussian coeffs is based on "up" & "down" operators ....